Math 776 Graph Theory Lecture Notes
Extremal Graph Theory

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Theorem 1 (Mantel 1907) *The maximum number of edges in an n-vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$.*

Definition 1 *A graph G is* **H-free** *if G has no subgraph isomorphic to H.*

A triangle-free graph is C_3 free.

Example 1 *Bipartite graphs are C_3 free since they contain no odd cycles.*

Proof of Theorem 1: Let G be a triangle-free graph. Let x be a vertex of G with maximum degree. Let $k = d(x)$ be the degree of x. Let $N(x)$ be the set of neighbors of x. Since G is triangle-free, there are no edges whose endpoints are both in $N(x)$. $\overline{N(x)}$ forms a vertex cover. The number of edges $e(G)$ satisfies

$$\begin{aligned} e(G) &\leq \sum_{y \in \overline{N(x)}} d(y) \\ &\leq \sum_{y \in \overline{N(x)}} k \\ &= k \cdot |\overline{N(x)}| \\ &= k(n-k) \\ &\leq \left(\frac{k+(n-k)}{2}\right)^2 \\ &= \frac{n^2}{4} \end{aligned}$$

Since $e(G)$ is an integer,

$$e(G) \leq \lfloor \frac{n^2}{4} \rfloor.$$

This upper bound is reachable by

$$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}.$$

$$\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$$

Case 1: n is even

$$n = 2k$$

So,

$$\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = k^2$$

and

$$\lfloor \frac{n^2}{4} \rfloor = k^2$$

Case 2: n is odd

$$n = 2k + 1$$

So,

$$\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = k(k+1) = k^2 + k$$

and

$$\lfloor \frac{n^2}{4} \rfloor = \lfloor \frac{(2k+1)^2}{4} \rfloor = \lfloor k^2 + k + \frac{1}{4} \rfloor = k^2 + k$$

□

Definition 2 *A* **Turán Graph** $T_{n,r}$ *is a complete r-partite graph with n vertices whose partite sets differ in size by at most 1.*

Theorem 2 (Turán, 1941) *Among the n-vertex simple graphs with no $r+1$ clique, $T_{n,r}$ has the maximum number of edges.*

Lemma 1 *Among simple r-partite graphs with n-vertices, the Turán graph is the unique graph with the most edges.*

Proof of Lemma: Let G be a simple r-partite graph with the most edges.

1. G must be a complete r-partite graph.

2. Any two partite sets of G differ in size by at most one.

Prove 2. by contradiction. Say $|v_1| \leq |v_2| - 2$. Pick any vertex in v_2 and move it up into v_1. Call this graph G'.

$$\begin{aligned} e(G') &= e(G) - dx^{old} + dx^{new} \\ &= e(G) - (n - |v_2|) + (n - |v_1| - 1) \\ &= e(G) + |v_2| - |v_1| - 1 \\ &> e(G) \end{aligned}$$

Contradiction.
Say $n = ar + b$, $0 \leq b \leq r$. So, n=$\underbrace{(a+1) + ... + (a+1)}_{b} + \underbrace{a + ... + a}_{r-b}$.
Hence, $T_{n,r}$ has the most edges among r-partite simple graphs. □

Proof of Turán Theorem:
Claim: Among graphs with no $r+1$ clique, the maximum is achieved by an r-partite graph. For each G, construct H satisfies:

1. H is r-partite.
2. $e(G) \leq e(H)$.

Prove by induction on r.

Initial Case:
(r=1): G has no edges, i.e. G is K_2 free. This case is trivial.
(r=2): Mantel's Theorem

Inductive Hypothesis:
Suppose the claim is true for r. For $r+1$, let G be a graph containing no K_{r+2}. Let x be a vertex with maximum degree k. The induced subgraph on $N(x)$ contains no K_{r+1}. By inductive hypothesis, there is an r-partite graph H' with $e(G_{N(x)}) \leq e(H')$. Let H be the graph joining $\overline{N(x)}$ and H'.

1. H is an $r+1$ partite graph.
2. $$\begin{aligned} e(H) &= |\overline{N(x)}|k + e(H') \\ &\geq \sum_{y \in N(x)} dy + e(G_{N(x)}) \\ &\geq e(G). \end{aligned}$$

By the lemma, $T_{n,r}$ has the most edges. □

Definition 3 Turán number *$t(n, G)$ is the maximum number of edges that an n-vertex G-free simple graph can have.*

$$\begin{aligned} t(n, K_r) &= e(T_{n,r-1}) \\ &= \binom{n}{2} - b\binom{a+1}{2} - (r-1-b)\binom{a}{2} \end{aligned}$$

When b=0,

$$\begin{aligned} t(n, K_r) &= \binom{n}{2} - (r-1)\binom{\frac{n}{r-1}}{2} \\ &\approx (1 - \frac{1}{r-1})\binom{n}{2} + l.o.t. \\ &= (1 - \frac{1}{r-1})\binom{n}{2} + O(n^2) \end{aligned}$$

l.o.t = lower order term

What is $\lim_{n\to\infty} \frac{t(n,G)}{\binom{n}{2}}$?

Theorem 3 (Erdös-Simonovits-Stone 1946) *For any* G, $\lim_{n\to\infty} \frac{t(n,G)}{\binom{n}{2}} = 1 - \frac{1}{\chi(G)-1}$

Theorem 4 (Erdös-Stone) *For all integers* $r \geq 2$ *and* $s \geq 1$ *and every* $\varepsilon > 0$, *there exists an integer* n_0 *such that every graph with* $n \geq n_0$ *vertices and at least* $e(T_{n,r-1}) + \varepsilon n^2$ *edges contains* $K_r^{(s)}$ *as a subgraph.*

For any graph H, we define $H^{(s)}$ as follows:

1. $V(H^{(s)}) = \{V(H) \times [s]\}$
2. $E(H^{(s)}) = \{(u,i) \sim (v,j)\text{: if } uv \in E(H)\}$

$H^{(s)}$ is called the blow-up graph.
$K_r^{(s)} = T_{rs,r}$

Let G be a graph with $\chi(G) = r$ (means G is r-colorable). We can color $V(G)$ so that the endpoints of each edge receive different colors. We can see $G \subseteq K_r^{(s)}$ for some s.

$$t(n,G) \geq (1 - \frac{1}{r-1})\binom{n}{2} + O(n^2)$$

Turán graph $T_{n,r-1}$ is G-free and has $(1 - \frac{1}{r-1})\binom{n}{2} + O(n^2)$ edges. Because $G \subseteq T_{n,r-1}$, then $\chi(G) \leq r-1$

$$t(n,G) \leq (1 - \tfrac{1}{r-1})\tbinom{n}{2} + \varepsilon n^2 \text{ for any } \varepsilon > 0.$$

$$1 - \frac{1}{r-1} \leq \frac{t(n,G)}{\binom{n}{2}} \leq 1 - \frac{1}{r-1} + \varepsilon$$

Szemerédi's Regularity Lemma $\Rightarrow$ Blow-up Lemma $\Rightarrow$ Erdös-Stone Theorem
$\Rightarrow$ Erdös-Simonovits-Stone Theorem

Lemma 2 (Szemerédi's Regularity Lemma) *For any* $\varepsilon > 0$, *and every integer* $m \geq 1$, *there exists an integer* M *such that every graph of order at least* m *admits an* ε*-regular partition* $\{v_0, v_1, ..., v_k\}$ *with* $m \leq k \leq M$.

Definition 4 *A,B are disjoint sets. The* **density** $d(A,B) = \frac{e(A,B)}{|A||B|}$.

Definition 5 *(A,B) is called* ε**-regular** *if for all* $X \subseteq A, Y \subseteq B$ *with* $|X| \geq \varepsilon|A|$ *and* $|Y| \geq \varepsilon|B|$,

$$|d(X,Y)| - d(A,B)| \leq \varepsilon.$$

(ε is a very small constant $0 < \varepsilon < 1$).

Definition 6 *A* **partition** $\{v_0, v_1, ..., v_n\}$ *of* $V(G)$ *is called* ε**-regular** *if*

1. $|v_0| \le \varepsilon|v|$*,* v_0*-exceptional set*

2. $|v_1| = ... = |v_k|$

3. All but at most εk^2 *of the pairs* (v_i, v_j) *with* $1 \le i < j \le k$ *are* ε*-regular.*

Lemma 3 *Suppose* (A, B) *is* ε*-regular. Then for any* $Y \subseteq B$ *with* $|Y| \ge \varepsilon|B|$*, then all but* $\varepsilon|A|$ *of the vertices in* A *have (each) at least* $(d-\varepsilon)|Y|$ *neighbors in* Y*.*

Proof of Lemma: Let $X = \{v \in A : |N(v) \cap Y \le (d-\varepsilon)|Y|\}$. If $|X| \ge \varepsilon|A|$,
$e(X, Y) \le |X|(d-\varepsilon)|Y|$
$d(X, Y) < d - \varepsilon$ Contradiction. □

Lemma 4 (Blow-Up) *For all* $d \in (0, 1]$ *and* $\triangle \ge 1$*, there exists an* $\varepsilon_0 > 0$ *with the following property: If* G *is any graph,* H *is a subgraph with* $\triangle(H) \le \triangle$*,* $S \in N$ *and* R *is any regularity graph of* G *with* $\varepsilon \le \varepsilon_0$*,* $l \ge \frac{s}{s_0}$ *then* $H \subseteq R_s \Rightarrow H \subseteq G$*. (*$\triangle$ *= maximum degree)*

Proof of Lemma: By regularity lemma,

1. $|v_1| = |v_2| = ... = |v_k| = l$
2. $|v_0| \le \varepsilon l$
3. All but εk^2 pairs of (v_i, v_j) are ε-regular.

For vertex $v_i \in V(H)$, we maintain a set of candidates, S_i. $v_i \to V_{f(i)}$ where $f(i)$ is the index of partition that v_i belongs to.

Initially, $S_i = V_{f(i)}$. For i from 1 to $|H|$ pick $v_i \in S_i$ then update S_j for all $v_j \in N_H(v_i)$.

$$S_j \leftarrow N_G(v_i) \cap S_j$$
$$|S_j^{new}| \ge (d-\varepsilon)|S_j^{old}|$$

For any time, $|S_j| \ge (d-\varepsilon)^{\triangle} l$. Choose ε such that $(d-\varepsilon)^{\triangle} \ge (\triangle+1)\varepsilon$. So, $|S_j| \ge (d-\varepsilon)^{\triangle} l \ge (\triangle+1)\varepsilon l$. Since $(\triangle+1)\varepsilon l \ge \triangle\varepsilon l$, the procedure of selecting v_i will not stop. We construct $H \subseteq G$. □

Theorem 5 (Erdös-Stone) *Suppose* G *has* $e(K_{r-1}^{(s)}) + \lambda n^2$ *edges then there exists* N *for* $n > N_\lambda$ G *contains* $K_r^{(s)}$*.*

Sketch of proof:

$$K_r^{(s)} \subseteq R^{(s)}$$
$$K_r \subseteq R' \subseteq R$$

We define R' as follows:

1. $v_i v_j \in E(R')$ if (v_i, v_j) are ε-regular and $d(v_i, v_j) \ge \lambda$.

We show R' has enough edges $e(R') > (1 - \frac{1}{r-1})\binom{k}{2}$.