Lecture 2. Turán problem 2

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1 Erdős-Simonovits-Stone theorem

We have seen that Turán theorem determines the extremal number for cliques and describes the unique extremal structure. The natural next step is what if we forbid general graphs other than cliques? We shall see in this section a satisfying answer for all non-bipartite graphs.

The seminal result of Erdős and Stone shows that the extremal function of a general graph is completely determined by another important graph parameter: the chromatic number. Recall that the *chromatic number* of a graph H, denoted by $\chi(H)$, is the minimum number of colours needed to colour $V(H)$ so that adjacent vertices do not receive the same colour.

Theorem 1.1 (Erdős-Simonovits-Stone 1946)**.** *Let H be an arbitrary graph, then*[1]

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\frac{n^2}{2}.$$

Remark 1.2. Erdős-Simonovits-Stone theorem gives the asymptotics of the extremal number for all non-biparite H; while for bipartite H, it only implies that $\mathrm{ex}(n,H) = o(n^2)$. In fact, it is known that extremal number for bipartite graphs is polynomially smaller, i.e. for any bipartite H, there exists $c = c_H$ such that

$$\mathrm{ex}(n,H) = O(n^{2-c}).$$

The proof of Erdős-Simonovits-Stone theorem proceeds by building a large $(\chi(H)-1)$-partite subgraph. Here, we shall present a conceptually simpler 'modern' proof using Szemerédi regularity lemma. Szemerédi regularity lemma is a fundamental tool in extremal graph theory. We will come back to it systematically later and defer the proof of the embedding lemma below till then.

The *edge density* of a graph G is $e(G)/\binom{|G|}{2}$. A *homomorphism* from H to F is an adjacency preserving map from $V(H)$ to $V(F)$, i.e. $\varphi : V(H) \to V(F)$ such that $uv \in E(H) \implies \varphi(u)\varphi(v) \in E(F)$; and we call F a *homomorphic image* of H. Note that $K_{\chi(H)}$ is a homomorphic image of H.

Lemma 1.3 (Embedding lemma)**.** *Let H be a graph, then the following holds for all sufficiently large n. For any n-vertex H-free graph G with edge density τ, there exists a (reduced) graph R, with $e(R) \geq (\tau - o(1))\binom{|R|}{2}$, containing no homomorphic image of H.*

Proof of Theorem 1.1. By the definition of chromatic number, the $(\chi(H)-1)$-partite Turán graph is H-free, yielding the lower bound.

For the upper bound, let G be an n-vertex H-free graph and R be the corresponding reduced graph obtained from Lemma 1.3. Then R is $K_{\chi(H)}$-free, and by Turán theorem, its edge density, hence also that of G, is at most $1 - \frac{1}{\chi(H)-1} + o(1)$ as desired. □

[1]The term $o(1)$ throughout should be understood as a quantity tending to zero as n tends to infinity.

2 Stability method

One standard technique in attacking an extremal problem is the so-called *stability method*. We have seen Erdős-Simonovits stability theorem. Such kind of stability statements are not only interesting on its own, but also helpful in obtaining exact results in extremal combinatorics.

Often time (but not always), we can tackle an extremal problem with the following three steps:

- Step 1. Obtain asymptotic result;
- Step 2. Obtain stability;
- Step 3. Use the stability to get exact result.

The stability method is refered to Steps 2 and 3. Sometimes, the stability statement in Step 2 can be derived by a more careful analysis of the proof for asymptotic result in Step 1, e.g. Erdős-Simonovits stability can be derived from Motzkin-Straus symmetrisation proof. We shall illustrate Step 3 via a baby application: determining the extremal number of pentagon C_5.

Theorem 2.1. *For large n, we have* $\mathrm{ex}(n, C_5) = \lfloor n^2/4 \rfloor$.

Note that, for C_5, Step 1 follows from Erdős-Simonovits-Stone Theorem 1.1: $\mathrm{ex}(n, C_5) = n^2/4 + o(n^2)$. The stability statement in Step 2 reads as follows. It also has a simple proof using Szemerédi regularity lemma and Erdős-Simonovits stability. We will cover it when studying the regularity lemma.

Lemma 2.2. *Let $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds for large n. Let G be an n-vertex C_5-free graph. If $e(G) \geq n^2/4 - \delta n^2$, then G can be made bipartite by deleting at most εn^2 edges.*

We now complete Step 3. The idea of stability method is to utilise the asymptotic structure of the extremal configuration from stability to show that there cannot be imperfection.

Proof of Theorem 2.1. Let G be an n-vertex extremal C_5-free graph. For the lower bound, as $T_2(n)$ is also C_5-free, extremality of G implies $e(G) \geq e(T_2(n)) = \lfloor n^2/4 \rfloor$.

For the upper bound, we first reduce it to graphs with high minimum degree.

Claim 2.3. *There is a subgraph $G' \subseteq G$ with $|G'| \geq n/2$ and $\delta(G') \geq (1/2 - \varepsilon)|G'|$.*

Proof of claim. Repeatedly remove low degree vertices from G to get a sequence of graphs $G_0 = G, G_2, \ldots$, with $V(G_i) = V(G_{i-1}) \setminus \{v_i\}$ and $d_{G_{i-1}}(v_i) < (1/2 - \varepsilon)|G_{i-1}|$, for all $i \in \mathbb{N}$. This process must terminate before dropping down to $n/2$, for otherwise it can be checked that we would obtain an $n/2$-vertex subgraph $H \subseteq G$ with $e(H) \geq (1/4 + \Omega(\varepsilon^2))|H|^2$. This contradicts Theorem 1.1, as $H \subseteq G$ is C_5-free. ■

By the above claim and passing to a subgraph if necessary, we may assume that

$$\delta(G) \geq (1/2 - \varepsilon)n.$$

Let $V(G) = X \cup Y$ be a max-cut of G.[2] By Lemma 2.2, we have

$$e(G[X]) + e(G[Y]) \leq \varepsilon n^2.$$

[2] A max-cut of a graph G is a bipartition $V(G) = X \cup Y$ that maximises the number of *cross edges*, i.e. edges between the two partite sets X and Y. An important property of a max-cut, which we shall use shortly, is that every vertex in one part, say X, has as many neighbours in the other part Y than in its own part X, since otherwise moving this vertex from X to Y would increase the number of cross edges, contradicting to the fact that $X \cup Y$ is a max-cut.

Consequently, this max-cut is almost balanced, i.e.

$$|X|, |Y| = n/2 \pm 2\sqrt{\varepsilon}n.$$

Indeed, otherwise $e(G) \leq |X||Y| + e(G[X]) + e(G[Y]) < n^2/4$, a contradiction. We shall show that there is no edge inside X or Y, and so G is bipartite, which together with the extremality of G implies that G has to be $T_2(n)$, as $T_2(n)$ has the maximum size among all bipartite graphs.

To get rid of the imperfections (edges in X and Y), we first show that the inner degree is $o(n)$, i.e.

$$\Delta(G[X]), \Delta(G[Y]) \leq 2\sqrt{\varepsilon}n.$$

Suppose otherwise that there is some $v \in X$ with $d(v, X) \geq 2\sqrt{\varepsilon}n$.[3] As $X \cup Y$ is a max-cut, $d(v, Y) \geq d(v, X) \geq 2\sqrt{\varepsilon}n$. Note that as G is C_5-free, the bipartite graph induced between $X_H := N(v, X)$ and $Y_H := N(v, Y)$ in G is P_4-free, thus having only $O(n)$ edges. Then for large n, the number of missing edges in $G[X, Y]$[4] is at least $|X_H||Y_H| - O(n) \geq 3\varepsilon n^2$. So again $e(G) \leq |X||Y| - 3\varepsilon n^2 + e(G[X]) + e(G[Y]) < n^2/4$, a contradiction. With the additional information that inner degree is sublinear, we are now ready to show that there is not even a tiny bit of imperfection, i.e. not a single edge is allowed in X or Y.

Suppose uv is an edge in X. Let w be a third vertex in X. Using that $\Delta(G[X]) \leq 2\sqrt{\varepsilon}n$, $|X|, |Y| = n/2 \pm 2\sqrt{\varepsilon}n$ and $\delta(G) \geq (1/2 - \varepsilon)n$, we see that the common neighbourhood of u, v, w contains almost the entire set Y: $|N(u) \cap N(v) \cap N(w) \cap Y| \geq (1 - 10\sqrt{\varepsilon})|Y|$. Then two such common neighbours in Y together with u, v, w induces a copy of C_5, a contradiction. This completes the proof. □

3 C_4-free graphs and Sidon sets

We have seen a satisfying asymptotic solution in Erdős-Simonovits-Stone theorem for Turán problem when the forbidden graph is non-bipartite; and commented that for bipartite forbidden graph H, the extremal number is polynomially small, that is,

$$\mathrm{ex}(n, H) = O(n^{2-c_H}). \tag{1}$$

We will now turn to bipartite Turán problem and present some classical and some recent results. In general, the bipartite Turán problem is much less understood. There are many open problems. For more comprehensive account of this topic, we refer the readers to a survey of Füredi and Simonovits[5].

The earliest such result was by Erdős, who studied the extremal problem for 4-cycle C_4 when he worked on Sidon sets. Let us first obtain an upper bound via a double-counting argument.

Theorem 3.1. $\mathrm{ex}(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}) = (\frac{1}{2} + o(1))n^{3/2}$.

Proof. Let G be an n-vertex C_4-free graph. We count cherries $K_{1,2}$ in two different ways. We can get a cherry by picking a middle vertex v and then two of its neighbours. Then, as $f(x) = \binom{x}{2}$ is convex, using Jensen's inequality $\mathsf{E}(f(X)) \geq f(\mathsf{E}X)$ and writing $m = e(G)$, we see that the number of cherries is

$$\sum_{v \in V(G)} \binom{d(v)}{2} \geq n \binom{\frac{1}{n}\sum d(v)}{2} = n \binom{\frac{1}{n}2e(G)}{2} = \frac{2m^2}{n} - m.$$

On the other hand, the number of cherries must be at most $\binom{n}{2}$, for otherwise we get two $K_{1,2}$s with different middle vertices and the same end vertices, yielding a copy of C_4, a contradiction. Thus, solving $\frac{2m^2}{n} - m \leq \binom{n}{2}$ gives the desired bound. □

[3]We write $N(v, X) := N(v) \cap X$ for the set of neighbours of v in X, and $d(v, X) = |N(v, X)|$ for the degree of v in X.

[4]We write $G[X, Y]$ for the bipartite graph induced between X and Y in G.

[5]The history of degenerate (bipartite) extremal graph problems, arXiv:1306.5167.

Note that $C_4 = K_{2,2}$. By counting stars instead of cherries, we can get the following upper bound for complete bipartite graph $K_{s,t}$. We leave the proof as an exercise.

Theorem 3.2 (Kövari-Sós-Turán)**.** *Let $s, t \in \mathbb{N}$ with $s \leq t$. Then $ex(n, K_{s,t}) \leq tn^{2-1/s}$.*

In particular, this implies (1) as every bipartite graph H is a subgraph of $K_{s,t}$ for some s, t. It is a major open problem to determine the order of magnitude for $\text{ex}(n, K_{s,t})$. Matching lower bounds are known only when $s = 2, 3$ (Erdős-Rényi-Sós, Brown) and when $t \geq (s-1)! + 1$ (Norm graph by Alon, Rónyai and Szabó improving previous ones by Kollár, Rónyai and Szabó). The simplest unknown case is $K_{4,4}$, where the best lower bound comes from extremal $K_{3,3}$-free graphs.

Open problem 3.3. *Improve $\Omega(n^{5/3}) \leq \text{ex}(n, K_{4,4}) \leq O(n^{7/4})$.*

Let us see the connection between C_4-free graphs and Sidon sets, from which we can see that $\text{ex}(n, C_4) = \Theta(n^{3/2})$.

Exercise 3.4. A set $S = \{a_1, \ldots, a_k\} \subseteq \mathbb{N}$ is a *Sidon set* if all pairwise sums $a_i + a_j$, $i \leq j$, are distinct. In other words, there is no non-trivial solution to $a + b = c + d$.[6] It is known that the largest Sidon set in $[n]$ has size $(1 + o(1))\sqrt{n}$.

- Prove the weaker upper bound that every Sidon set in $[n]$ has size at most $2\sqrt{n}$.
- Use a Sidon set to construct an n-vertex C_4-free graph with $\Omega(n^{3/2})$ edges.

We now present a construction of polarity graph that shows that the bound in Theorem 3.1 is asymptotically tight.

Theorem 3.5 (Erdős-Rényi-Sós)**.** $\text{ex}(n, C_4) \geq (\frac{1}{2} - o(1))n^{3/2}$.

Proof. For large enough n, it is known that there exists a prime p between $(1 - o(1))\sqrt{n+1}$ and $\sqrt{n+1}$. Consider the following graph G:

$$V(G) = \mathbb{F}_p^2 \setminus \{(0,0)\} \text{ and } E(G) = \{(a,b)(x,y) : ax + by = 1\}.$$

Delete loops. Take two distinct vertices $\boldsymbol{v}_i = (a_i, b_i)$, $i \in [2]$. For each $i \in [2]$, $N(\boldsymbol{v}_i)$ consists of vertices (x, y) satisfying $a_i x + b_i y = 1$, which is a line in $\mathbb{F}_p^2$. As $\boldsymbol{v}_1, \boldsymbol{v}_2$ are distinct, $N(\boldsymbol{v}_1)$ and $N(\boldsymbol{v}_2)$ are distinct lines, and so intersecting at at most one point. That is, $\boldsymbol{v}_1, \boldsymbol{v}_2$ have at most one common neighbour, hence G is C_4-free.

Note that every vertex has degree p or $p - 1$ (at most one loop incident to a vertex), so $e(G)\frac{1}{2}(p-1)(p^2-1) \geq (\frac{1}{2} - o(1))n^{3/2}$. □

There is a related extremal problem asked by Fischer and Matousek, for which finding new algebraic constructions of dense C_4-free graphs seems like a right way to go.

Open problem 3.6. *Let G be an $n \times n \times n$-vertex 3-partite graph. If between any two partite sets, the induced bipartite graph is C_4-free, then how many triangles can G have?*

An upper bound of $n^{7/4}$ can be easily obtained via Cauchy-Schwarz. The best lower bound is $n^{5/3}$ due to Coulter-Matthews-Timmons.

[6] By trivial solution, we mean $\{a, b\} = \{c, d\}$.