Theorem 15.4. *For* $X \sim \text{Geometric}(p)$, *we have* $\mathbb{E}[X] = \frac{1}{p}$.

Proof. The key observation is that for a geometric random variable X,

$$\mathbb{P}[X \geq i] = (1-p)^{i-1} \quad \text{for } i = 1, 2, \ldots. \tag{5}$$

We can obtain this simply by summing $\mathbb{P}[X = j]$ for $j \geq i$. Another way of seeing this is to note that the event "$X \geq i$" means at least i tosses are required. This is equivalent to saying that the first $i-1$ tosses are all Tails, and the probability of this event is precisely $(1-p)^{i-1}$. Now, plugging equation (5) into Theorem 15.3, we get

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \frac{1}{1-(1-p)} = \frac{1}{p},$$

where we have used the formula for geometric series. □