Bertrand's ballot problem: Let $a \geq b$.
In an election,

Candidate A receives $a$ votes

Candidate B receives $b$ votes

Suppose that the votes are counted in a random order.

What is the probability that A never trails?

Solution:

Define

$$\mathcal{S} = \left\{ \begin{array}{l} \text{strings of size } a+b \text{ with} \\ a \text{ letters A and } b \text{ letters B} \end{array} \right\}$$

Q/ What is $|\mathcal{S}|$? $\binom{a+b}{b}$

For a string $s \in \mathcal{S}$, let

$s_i(a)$ = # letters a until position $i$

$s_i(b)$ = # letters $b$ until position $i$

Goal : Calculate the size of

$$P = \{ s \in \mathcal{S} : s_i(a) \geq s_i(b) \ \forall i \}$$

It turns out to be easier to calculate

the size of $|\mathcal{S} \setminus P|$.

Obs : If $s \in \mathcal{S} \setminus P$, then $\exists$ $K$ minimum

such that $s_i(a) = K$ and $s_i(b) = K+1$

for some $i$ (Not hard : $i = 2K+1$)

To calculate the size of $|\mathcal{S} \setminus P|$,

we create a bijection between

$\mathcal{S} \setminus P$ and another set which is

easier to count.

Let $f : \mathcal{S} \setminus P \to P$ be defined as

$$(f(s))_i = \begin{cases} s_i & \text{if } i \leq 2K+1 \\ A & \text{if } i \geq 2K+2 \text{ and } s_i = B \\ B & \text{if } i \geq 2K+2 \text{ and } s_i = A \end{cases}$$

Q/ How many A's and B's are there in $f(s)$?

$K$ A's, $K+1$ B's $\quad | \quad$ $a-K$ A's $\quad$ $b-K-1$ B's

$S \quad \circ \quad \circ \quad \circ \quad \circ \; \cdots \; \circ \; \cdots \; - \; - \; \circ$

$2K+1$

$\Rightarrow f(S)$ has $K + b - K - 1 = b - 1$ A's

$K + 1 + a - K = a + 1$ B's

In particular,

$\# B\text{'s in } f(S) \geq \# A\text{'s in } f(S)$

$\Rightarrow \exists\, K$ such that $(f(S))_{2K+1}(a) = K$

$(f(S))_{2K+1}(b) = K+1$

( we always take $k$ to be the minimum element with this property)

$\Rightarrow$ $f$ is indeed a bijection.

<u>Prob</u>:

$$\frac{\binom{a+b}{a} - \binom{a+b}{a+1}}{\binom{a+b}{a}}$$