

Bertrand's ballot problem: Let $a \geq b$.

In an election,

Candidate A receives a votes

Candidate B receives b votes

Suppose that the votes are counted
in a random order.

What is the probability that A
never trails?

Solution :

Define

$S = \{ \text{strings of size } a+b \text{ with} \}$
 $\quad \quad \quad a \text{ letters A and } b \text{ letters B} \}$

Q/ What is $|S|$? $\binom{a+b}{b}$

For a string $s \in S$, let

$s_i(a) = \# \text{ letters a until position } i$

$s_i(b) = \# \text{ letters } b \text{ until position } i$

Goal : Calculate the size of

$$P = \{ s \in S : s_i(a) \geq s_i(b) \forall i \}$$

It turns out to be easier to calculate the size of $|S \setminus P|$.

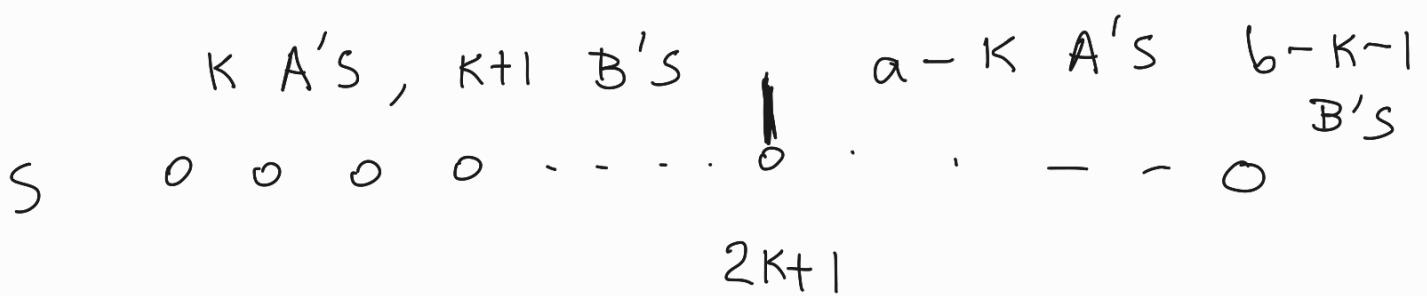
Obs : If $s \in S \setminus P$, then $\exists k \text{ minimum such that } s_i(a) = k \text{ and } s_i(b) = k+1$ for some i (Not hard : $i = 2k+1$)

To calculate the size of $|S \setminus P|$, we create a bijection between $S \setminus P$ and another set which is easier to count.

Let $f : S \setminus P \rightarrow P$ be defined as

$$(f(s))_i = \begin{cases} s_i & \text{if } i \leq 2k+1 \\ A & \text{if } i \geq 2k+2 \text{ and } s_i = B \\ B & \text{if } i \geq 2k+2 \text{ and } s_i = A \end{cases}$$

Q / How many A's and B's are there in $f(s)$?



$$\Rightarrow f(s) \text{ has } k + b - k - 1 = b - 1 \quad \text{A's}$$

$$k + 1 + a - k = a + 1 \quad \text{B's}$$

In particular,

$$\# \text{B's in } f(s) \geq \# \text{A's in } f(s)$$

$$\Rightarrow \exists k \text{ such that } (f(s))_{2k+1}(a) = k$$

$$(f(s))_{2k+1}(b) = k+1$$

(we always take κ to be the minimum element with this property)

$\Rightarrow f$ is indeed a bijection.

$$\text{Prob} : \frac{\binom{a+b}{a} - \binom{a+b}{a+1}}{\binom{a+b}{a}}$$

