We can also negate a quantified expression.

For example, as $\forall x P(x)$ is a proposition, we can consider $\neg \forall x P(x)$.

In this case, note that

$$\neg \forall x P(x) \equiv \exists x \neg P(x).$$

In general, we have the following De Morgan's laws for quantifiers:

Negation	Equivalent statement	When is negation true?
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There exists an x for which $P(x)$ is false.

Obs: in the book you might see a notation such as

$$\forall x<0\, \underbrace{(x^2>0)}_{}$$

this expression is the predicate.

Nested quantifiers:

Consider an expression such as $\forall x \exists y\, (x+y=0)$.

This is the same as

$\forall x\, Q(x)$, where $Q(x)$ is $\exists y\, (x+y=0)$.

The order of quantifiers is important.

Example:

Let $Q(x,y)$ denote "$x+y=0$".

The truth value of the quantification $\forall x \, \exists y \, Q(x,y)$ is true.

The truth value of the quantification $\exists y \, \forall x \, Q(x,y)$ is false.

Elementary methods of proof:

Direct proof: Assume P, follow logical deductions, conclude Q.

Contrapositive: Assume $\neg Q$, follow deduction to conclude $\neg P$.

Method of contradiction: Assume that P and $\neg Q$, follow deductions, obtain a contradiction.

Example of a direct proof:

Lemma: Let $y_1, \ldots, y_n \in \mathbb{R}$. Then, some number is as large as the average.

proof: Let y^* be the largest number in the set. Then,

$$y^* \cdot n \geq \sum_{i=1}^{n} y_i \text{. This implies } y^* \geq \frac{\sum_{i=1}^{n} y_i}{n}.$$

$\blacksquare$

Example of a

of a proof by contradiction:

Definition: A number p is prime if $p > 1$ and if p is only divisible by 1 and itself.

Example: 2 is prime, 3 is prime, 4 is not prime.

Theorem: There are infinitely many primes.

Proof: suppose for contradiction that there is a finite list of primes, say $p_1, p_2, \cdots, p_n$.
Consider the number

$$N = p_1 p_2 \cdots p_n + 1.$$

N is not divisible by neither p_1 nor p_2 nor $\cdots$ p_n (it leaves 1 as remainder). Therefore, either N is prime or N is divisible by a prime not in the list.
This is a contradiction! $\blacksquare$

Example of a proof by the contrapositive method:

Lemma: Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = mx + b$, where $m \neq 0$ and $b \in \mathbb{R}$. If $x \neq y$, then $f(x) \neq f(y)$.

proof: The contrapositive is: if $f(x) = f(y)$ then $x = y$.
Note that if $f(x) = f(y)$, then $mx + b = my + b$,
This implies $x = y$.

Algebraic manipulations:

Example ①: Find the solutions of the equation $x^2 = 5x$.

First, suppose that we have a solution $x \neq 0$.

Then, dividing both sides by x gives $x = 5$.

Now, note that $x = 0$ is also a solution.

Example ②: Find the flaw in the following algebraic manipulations: Let x, y be real numbers.

$$\begin{aligned} x = y \Rightarrow\ & x^2 = xy \\ \Rightarrow\ & x^2 - y^2 = xy - y^2 \\ \Rightarrow\ & (x+y)(x-y) = y(x-y) \\ \Rightarrow\ & x + y = y. \end{aligned}$$

This step cannot be done, as $x - y = 0$.

In the special case $x = y = 1$ we obtain $2 = 1$.

From now on we denote $P \Leftrightarrow Q$ when the propositions P and Q are equivalent.

Obs: $Q \Rightarrow P$ is called the converse of $P \Rightarrow Q$.

Do not confuse converse with contrapositive.

Proof strategies

Look for invariants.

Problem: Suppose there are 1 to 1000 numbers written on a paper. At each step, we select any two numbers (randomly) and replace them with their difference.

Is it possible to have 243 in the end as the only remaining number?

Solution: Let S_i be the total sum after i steps, with S_0 being the initial sum.

Let $b_i > a_i$ be the numbers chosen in step i.

Then, $S_{i+1} = S_i - a_i - b_i + (a_i + b_i)$

$$= S_i - 2b_i$$

This implies that the difference $S_i - S_{i+1}$ is always even.

Our initial sum is $1 + \dots + 1000 = \frac{1001 \cdot 1000}{5}$, which is also an even number.

Therefore, we cannot reach an odd number, as we always have an even sum in each step.