Definition: A compound proposition is an expression formed from propositional variables using logical operators.

Definition (Satisfiable): we say that a compound proposition is satisfiable if there exists an assignment of truth values to its variables that makes it true. When no such assignment exists, we call it unsatisfiable.

Definition (Solution): A solution is an assignment of truth values that makes a compound proposition true.

Exercise: Determine weather the expression

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

is satisfiable.

Solution: For this expression to be satisfiable, we need

① $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ to be true and

② $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ to be true.

For the 1st to be true we need either p, q and r to be true, or p, q, r to be false.

For the 2nd to be true, we need that at least one of the statements [illegible] is [illegible] at least of the statements p, q and r is true and at least one is false.

It follows that our expression is always false.

Propositional logic cannot adequately express the meaning of all statements in mathematics.

For example, suppose that we know that

"Every computer in UIUC is functioning properly".

No rules of propositional logic allows us to conclude that

" M3 is functioning properly"

where M3 is a computer in UIUC.

That's why we now introduce <u>predicate logic</u>.

The assertions

"$x > 3$", "$x = y + 3$", "computer x is under attack"

can be divided into <u>subject</u> (variable x) and <u>predicate</u> (property that the subject can have).

We can denote the statetement

"x is greater tha 3" by $P(x)$, where P denotes the predicate (is greater than x) and x is the variable.

P(x) is also said to be the value of the <u>propositional function</u> P at x.

Once a value has been assigned to x, P(x) becomes a proposition.

Let A(x) = "computer x is functioning properly".
Let M3 be a computer in UIUC.
Suppose that all computers in UIUC are functioning properly.
Then, A(M3) is true.

<u>Definition</u>: A assertion of the form $P(x_1, \ldots, x_n)$ is called an n-ary predicate.

<u>Quantifiers</u>

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value.
However, there is another way called <u>quantification</u>

<u>Definition</u>: The universal quantification of P(x) is the statement
"P(x) for all values of x in the domain".
<u>Notation</u>: $\forall x \, P(x)$. (read: for every x P x)
$\forall$ is called universal quantifier.

An element for which $P(x)$ is false is called a counterexample of $\forall x P(x)$.

Examples:

① Let $P(x)$ be the statement "$x+1 > x$".
what is the truth value of the quantification $\forall x P(x)$ when the domain is $\mathbb{R}$?

Truth value of $\forall x P(x)$ is true.

② $Q(x) =$ "$x < 2$".

Find truth value of $\forall x Q(x)$ when the domain is $\mathbb{R}$.

Truth value $=$ False.

Obs: when the domain can be listed as $x_1, \dots, x_n$.

The truth value of $\forall x P(x)$ is the same as $P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$.

Definition: The existential quantification of $P(x)$ is the proposition
"There exists an element x in the domain such that $P(x)$"

Notation: $\exists x P(x)$ (read: there exists x such that $P(x)$).

$\exists$ is called existential quantifier.

Examples:

① $P(x) =$ " $x > 3$ " with domain $\mathbb{R}$.

Truth value of $\exists x P(x)$ is true

② $P(x) =$ " $x = x+1$ " with domain $\mathbb{R}$.

Truth value of $\exists x P(x)$ is false.

When the elements of the domain can be listed $x_1, \dots, x_n$. The truth value of $\exists x P(x)$ is the same as $P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)$.

Other types of quantifiers :

- $\exists! x P(x)$ = there exists an unique x such that $P(x)$.
- There are no more than 3
- There are at least 100

$\vdots$

In the same way that we have equivalences of compound propositions, we can extend this notion to expressions involving predicates and quantifiers.

Definition : Statements involving predicates and quantifiers are logically equivalent if and only if they that the same truth value no matter which predicates and domains are substituted.

Notation : $S \equiv T$

Example: $\forall x (P(x) \wedge Q(x)) \equiv (\forall x P(x)) \wedge (\forall x Q(x))$