We now introduce a new method of proof, called method of descent. We shall use this method to show that $\nexists\, x \in \mathbb{Q}$ st $x^2 = 2$.

Method of descent:

Suppose that we want to show that $S = \mathbb{N}$.
We suppose for contradiction that $S \neq \mathbb{N}$.
Then, the set $S^c := \mathbb{N} \setminus S$ has a minimum element (well-ordering principle).
Let $n_0 = \min S^c$. If we show that $\exists\ k < n_0$ st $S^c \ni k$, we obtain a contradiction.

Theorem: There are no solutions for $x^2 = 2$ in the rationals.

proof: Suppose for contradiction that m/n is a solution.

We have $1 < \frac{m}{n} < 2$. $\Rightarrow$ $n < m < 2n$.

$$\Rightarrow \quad 0 < m - n < n$$

Note that $2n^2 = m^2$, and hence

$$\frac{2n-m}{m-n} = \frac{n(2n-m)}{n(m-n)} = \frac{2n^2 - mn}{n(m-n)} = \frac{m^2 - mn}{n(m-n)} = \frac{m(m-n)}{n(m-n)}$$

$$= \frac{m}{n}$$

$\blacksquare$

The real numbers

Sequences

A sequence is a function $X : \mathbb{N} \to S$

which associates to each natural number n an element of S.

For now $S = \mathbb{Q}$.

<u>Obs</u>: $(x_1, x_2, \ldots) \neq \{x_1, x_2, \ldots\}$

<u>Definition</u>: We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ converges to a limit $L \in \mathbb{Q}$ if $\forall\, M > 0\ \exists\, N_0 \in \mathbb{N}$ such that

$$|x_n - L| < \frac{1}{M}$$

for all $n \geq N_0$.

<u>Problem</u>: Show that the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = \frac{1}{n}$ converges to 0.

<u>Problem</u>: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence defined as $a_1 = 1$ and $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$. Show that this sequence cannot tend to a rational limit.

The elements in the sequence are getting closer but the limit does not exist. We want to force this limit to exist. To do so, we do what we call "completion of $\mathbb{Q}$".

<u>Definition</u>: A Cauchy sequence is a sequence $(x_1, x_2, x_3, \dots)$ such that $\forall\ \varepsilon \in \mathbb{Q}_{>0}$ $\exists\ N = N(\varepsilon) \in \mathbb{N}$ such that

$|x_n - x_m| < \varepsilon \quad \forall\, m, n \geq N.$

Example:

- $(1, 1, \dots)$
- A subsequence of a Cauchy sequence

Example of a non-Cauchy sequence:

- $(1, 2, 3, \dots)$
- $(H_1, H_2, H_3, \dots)$, where $H_i := 1 + \frac{1}{2} + \dots + \frac{1}{i}$

<u>Lemma</u>: Every Cauchy sequence is bounded.

<u>proof</u>: Intuition is to write $x_n = x_n - x_N + x_N$, for some fixed N.

all of our sequence is living here.