<u>Definition</u>: The set $\mathbb{Q}$ is equal to the collection of all equivalence classes of $\mathbb{Z} \times \mathbb{N}$ under the following equivalence relation:

$(a, b) \sim (c, d) \iff b \cdot c = a \cdot d$.

We denote by a/b the equivalence class of the pair (a, b).

<u>Notation</u>: $\mathbb{Q} = \mathbb{Z} \times \mathbb{N} / \sim$

Again, we can formally define sum, product and order.

<u>Sum</u>: Let $X, Y \in \mathbb{Q}$ be two equivalence classes.

Take $(a, b) \in X$ and $(c, d) \in Y$.

Set $X + Y = Z$, where Z is the class of $(ad + bc, bd)$.

<u>Lemma</u>: Addition on $\mathbb{Q}$ is well-defined.

<u>proof</u>: If $(a,b) \sim (w, x)$ and $(c,d) \sim (y, z)$, then we have $bw = ax$ and $cz = dy$. Thus,

$$
\begin{aligned}
(a,b) + (c,d) &= (ad + bc, bd) \\
&= (xzad + xzbc, xzbd) \\
&= (bwdz + bxdy, bdxz) \\
&= (wz + xy, xz) \\
&= (w, x) + (y, z).
\end{aligned}
$$

We also define multiplication by setting:

$X \cdot Y = ((ac, bd))$, where $(a, b) \in X$ and $(c, d) \in Y$.

The rational numbers satisfy all the properties listed for the integers, along with the following new properties:

- Inverses $(\cdot)$: $\forall a \in \mathbb{Q}, a \neq 0, \exists$ unique $b \in \mathbb{Q}$ such that $a \cdot b = 1$.

- Archimedean property: $\forall\ p/q \in \mathbb{Q}$ with $p/q > 0$ $\exists\ n \in \mathbb{N}$ st

$$0 < \frac{1}{n} < p/q .$$

<u>proof of A.p</u> (assuming all good properties about $\mathbb{Q}$ and $<$):

$\frac{p}{q} \cdot 2q \geqslant 2p > 1$. Take $n = 2q$.

$\boxtimes$