The integers

Before defining $\mathbb{Z}$, we need the definition of equivalence classes.

Definition: A relation $\equiv$ on a set S is called an equivalence relation if it satisfies the following three properties:

1. Reflexivity: For any $x \in S$, we have $x \equiv x$
2. Symmetry: For any $x, y \in S$, if $x \equiv y$, then $y \equiv x$
3. Transitivity: For any $x, y, z \in S$, if $x \equiv y$ and $y \equiv z$, then $x \equiv z$.

Examples of equivalence relations: equality, remainder in the division by n.

Examples of non-equivalence relations: subset inclusion, less than

Definition (equivalence class): Take any set with an equivalence relation $\equiv$. For any element $x \in S$, we can define the equivalence class corresponding to x as the set $\{ s \in S : s \equiv x \}$.

Recall the Peano's axioms

Recall that the set of natural numbers $\mathbb{N}$ is characterized by the following axioms:

1. $\exists$ injective function $s : \mathbb{N} \to \mathbb{N}$.
2. $\exists$ an element in $\mathbb{N} \setminus s(\mathbb{N})$, which we call 1.
3. If $A \subseteq \mathbb{N}$ is such that $1 \in A$ and $s(A) \subseteq A$, then $A = \mathbb{N}$.

Definition: The set $\mathbb{Z}$ is equal to the collection of all equivalence classes of $\mathbb{N} \times \mathbb{N}$ under the following equivalence relation:
$(a,b) \sim (c,d) \iff a+d = c+b$.

We denote by $a-b$ the equivalence class of the pair (a,b).

Notation: $\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$.

The class of the pair $(1,1)$ has a special name: 0.

- Note that $(a,a) \sim (1,1)$ $\forall a \in \mathbb{N}$.
- $(a,b) \sim (a+k, b+k)$ $\forall a, b, k \in \mathbb{N}$:

$$a + (b+K) = b + (a+K).$$

- Each element of $\mathbb{Z}$ is a set, which is an equivalence class.

<u>How to add, multiply and order in $\mathbb{Z}$</u>:

<u>Sum</u>:

Let X and Y be two equivalence classes in $\mathbb{Z}$.

We define $X+Y$ to be the equivalence class containing (x_1+y_1, x_2+y_2), where $(x_1,x_2) \in X$ and $(y_1,y_2) \in Y$.

<u>Claim</u>: This choice does not depend on the choice of the representatives.

<u>Product</u>:

Let X and Y be two equivalence classes in $\mathbb{Z}$.

We define $X \cdot Y$ to be the equivalence class containing $(x_1y_1 + x_2y_2, x_2y_1 + x_1y_2)$, where $(x_1,x_2) \in X$ and $(y_1,y_2) \in Y$.

<u>Claim</u>: This choice does not depend on the choice of the representatives.

To prove this claim, we need the following lemma:

<u>Lemma</u>: Suppose that $(a,b) \sim (c,d)$ and $c > a$. Then, $\exists\, k \in \mathbb{N}$ such that $c = a + k$ and $d = b+k$.

With this lemma, the claim follows easily from the properties of $(\mathbb{N}, +, \cdot)$.

<u>Properties of $\mathbb{Z}$</u>:

<u>Closure</u>: $\forall a, b \in \mathbb{Z}$, $a+b \in \mathbb{Z}$, $ab \in \mathbb{Z}$

<u>Identity</u>: $\exists\, 0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ st $\forall a \in \mathbb{Z}$ we have $0 + a = a$ and $1 \cdot a = a$

Commutativity, associativity and distributivity.

Inverses $(+)$: $\forall a \in \mathbb{Z}\ \exists$ a unique $b \in \mathbb{Z}$ such that $a+b = 0$.

... these properties.

You can rigorously prove that $\mathbb{Z}$ satisfies the prop

We also have <u>order</u>:

Given two equivalence classes $x, y \in \mathbb{Z}$, we say that $x < y$ if $x_1 + y_2 < y_1 + x_2$, where $(x_1, x_2) \in X$ and $(y_1, y_2) \in Y$.

We can show that $\mathbb{Z}$ satisfies all the properties we know about order.

Moreover, $\mathbb{Z}$ satisfies two extra properties

- $\forall\, a \in \mathbb{Z}$, $0 \cdot a = 0$
- $\forall\, a \in \mathbb{Z}$, $(-a) = (-1) \cdot a$.
