Example: Describe the time complexity of the binary search in terms of the number of comparisons used (ignore the time to compute $\lfloor \frac{i+j}{2} \rfloor$).

Binary search algorithm

PROCEDURE binary search $(x \in \mathbb{R}, (a_1, \dots, a_n) \in \mathbb{R}$ in increasing order $)$

$i \leftarrow 1$ (i is the left endpoint of search interval)

$j \leftarrow n$ (j is the right endpoint of search interval)

WHILE $i < j$:

$m \leftarrow \lfloor \frac{i+j}{2} \rfloor$

If $x > a_m$, then $i \leftarrow m+1$

Else $j \leftarrow m$.

If $x = a_i$, then location $\leftarrow i$

Else location $\leftarrow 0$.

Return location.

At each iteration we make two comparisons:

$i < j$ and $x > a_m$.

As in each step we half the interval, we iterate $\lceil \log_2 n \rceil$ times in total.

We have one more comparison to exit the loop and one more outside the loop.

In total, $2\lceil \log_2(n) \rceil + 2 = O(\log n)$.

The type of complexity analysis done so far is a worst-case analysis. By the worse-case performance of an algorithm, we mean the largest number of operations needed to solve a problem in terms of the size of the input.

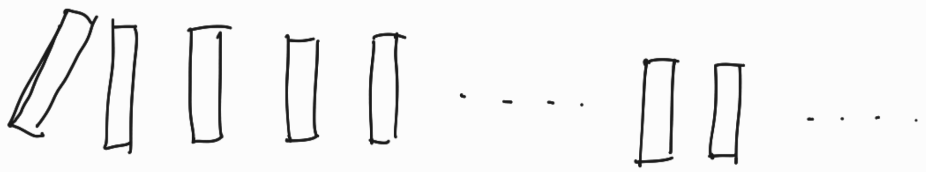
PRINCIPLE OF MATHEMATICAL INDUCTION.

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Basis step: We verify that $P(1)$ is true.

Inductive step: we show that the conditional statement $P(k) \Rightarrow P(k+1)$ is true for all positive integers.

Ways to remember how mathematical induction works:



$P(n)$ = domino n is knocked over

If $P(1)$ is true, then $P(2)$ is true.

In general, if $P(K)$ is true, then $P(K+1)$ is true.

This implies that if 1 is knocked over, then all the dominos are knocked over.

Problem: Show that if n is a positive integer, then $1+2+\ldots+n = \dfrac{n(n+1)}{2}$.

Solution:

Base of induction: The formula holds for $n=1$: $1 = \dfrac{1 \cdot 2}{2}$.

Inductive step: Assume that the formula holds for for an arbitrary $K \in \mathbb{N}$. That is,

$$1+2+\ldots+K = \frac{K\cdot(K+1)}{2}.$$

Now, we should show that it also holds for $K+1$:

$$1+\ldots+K+(K+1) \overset{\text{I.H.}}{=} \frac{K(K+1)}{2}+(K+1)$$

$$= \frac{K(K+1)+2(K+1)}{2}$$

$$= \frac{(K+2)(K+1)}{2}$$

$$= \frac{((K+1)+1)(K+1)}{2}$$

We have completed the basis step and the inductive step.

By mathematical induction it follows that the formula holds for all $n \in \mathbb{N}$.

Problem: Conjecture a formula for the sum of the first n positive odd integers. Prove your conjecture.

Solution: We can see that

$$1 = 1, \quad 1 + 3 = 4, \quad 1 + 3 + 5 = 9$$

Conjecture: Let $S(n)$ be the sum of the n-th first odd numbers. Then, $S(n) = n^2$.

~~proof of the conjecture:~~

Base of induction: it holds for $n = 1$

$$(S(1) = 1^2).$$

Induction step: suppose it holds for $K \in \mathbb{N}$.

We will prove it holds for $K+1$.

Let $p(K)$ be the Kth positive odd integer. Note that $p(K) = 2K - 1$.
(you can prove this by induction).

Now note that

$$S(K+1) = S(K) + p(K+1)$$

By the induction hypothesis, we have $S(K) = K^2$ and hence

$$S(K+1) = K^2 + 2(K+1) - 1$$

$S(K+1) =$ [illegible]

$$= K^2 + 2K + 1$$

$$= (K+1)^2$$

We have completed the basis step and the inductive step.

By mathematical induction it follows that the formula holds for all $n \in \mathbb{N}$.