# The halting problem

The halting problem is an example of a problem that cannot be solved by any procedure.

<u>Problem</u>: Determine weather there is a procedure that takes as an input a pair $(P, I)$, where $P$ is a program and $I$ is an input of $P$, and output weather $P$ will eventually stop when run with this input.

<u>Solution</u>: We will show that no such procedure exists. Suppose for contradiction that it does. Let us call it $H$.

Let $P$ be a program and $I$ be an input of $P$.

We have

$$\text{Output of } H(P,I) = \begin{cases} \text{"halt", if } P \text{ stops with input } I \\ \text{"loops forever", otherwise.} \end{cases}$$

A program can have itself as an input.

If this does not "make sense", the algorithm will output an error. $H$ will determine if this error is obtained in a finite amount of time or if it loops forever.

Let $K$ be the procedure defined as follows:

$K(P)$ will $\{$ "halt" if output of $H(P,P)$ is "loops forever"

K(P) will { "loops forever" otherwise.

Let us provide K as input to K.

If H(K, K) loops forever, then K(K) halts (by definition of K).

If H(K, K) halts, then K(K) loops forever (by definition of K).

But H(K, K) and K(K) must have the same answers. This is a contradiction. $\blacksquare$

## The growth of functions

In order to compare how long it takes for some algorithms to perform a given task, we introduce the big-O notation.

**Definition**: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be functions, where $A = \mathbb{Z}$ or $A = \mathbb{R}$.

We write $f(x) = O(g(x))$ if there are constants $C$ and $K$ such that

$$|f(x)| \leq C|g(x)|.$$

whenever $x \geq K$.

(read: $f(x)$ is big-oh of $g(x)$).

Meaning: f grows slower that some fixed multiple of g as x grows.

Obs: There might be many valves of C and K which witness the big-oh.

Example: Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 2x + 1$.
Show that $f(x) = O(x^2)$.

Lemma: Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions such that

$$f(x) = O(g(x)) \text{ and } g(x) = O(h(x)).$$

Then, $f(x) = O(h(x))$.

proof: $f(x) = O(g(x)) \Rightarrow \exists K_1, C_1$ such that

① $$|f(x)| \leq C_1 |g(x)| \quad \forall x \geq K_1$$

In the same way, $g(x) = O(h(x)) \Rightarrow \exists K_2, C_2$ such that

② $$|g(x)| \leq C_2 |h(x)| \quad \forall x \geq K_2.$$

From ① and ② we obtain that

$$|f(x)| \leq C_1 |g(x)| \leq C_1 \cdot C_2 |h(x)|$$

for every $x \geq \max\{K_1, K_2\}$.

Therefore, $C_1 \cdot C_2$ and $\max\{K_1, K_2\}$ are our constants which witness that $f(x) = O(h(x))$.

Definition: Let $f, g : A \to \mathbb{R}$ be functions, where $A = \mathbb{Z}$ or $A = \mathbb{R}$. We say that $f$ and $g$ have the same order if $f(x) = O(g(x))$ and $g(x) = O(f(x))$.

Lemma: If $f, g : A \to \mathbb{R}$ have the same order, then there exists $C$ and $K$ such that

$$|f(x)| \leq C|g(x)| \text{ and } |g(x)| \leq C|f(x)|$$

for every $x \geq K$.

proof: $f(x) = O(g(x)) \Rightarrow \exists K_1, C_1$ such that

(1) $$|f(x)| \leq C_1 |g(x)| \quad \forall x \geq K_1$$

In the same way, $g(x) = O(f(x)) \Rightarrow \exists K_2, C_2$ such that

(2) $$|g(x)| \leq C_2 |f(x)| \quad \forall x \geq K_2.$$

Take $C = \max\{C_1, C_2\}$ and $K = \max\{K_1, K_2\}$.

From (1) and (2) we obtain that

$$|f(x)| \leq C|g(x)| \text{ and } |g(x)| \leq C|f(x)|$$

for every $x \geq K$.

$\square$

Notation: We denote by $f(x) = \Theta(g(x))$ when $g$ and $f$ have the same order.

**Definition:** Let $f, g : A \to \mathbb{R}$ be functions, where $A = \mathbb{R}$ or $\mathbb{Z}$. We write $f(x) = \Omega(g(x))$ if there exists positive constants $C$ and $K$ such that

$$|f(x)| \geq C|g(x)|$$

whenever $x > K$.

**Observation:** $f(x) = \Omega(g(x))$ if and only if $g(x) = O(f(x))$