Problem ①: Use mathematical induction to show that

$$1+2+2^2+\dots+2^n = 2^{n+1}-1.$$

Solution: Observe that the formula holds for $n=1$: $1+2=2^{1+1}-1$.

Define

$$A=\{n\in\mathbb{N} : 1+2+2^2+\dots+2^n = 2^{n+1}-1\}$$

Note that $1\in A$. This is called the base of the induction.

Let $k\in A$ be an element in A. Next, we show that $k+1\in A$. This is called the inductive step.

If $k\in A$, then

$$1+2^2+\dots+2^k+2^{k+1} = 2^{k+1}-1+2^{k+1}.$$

here we used that $k\in A$.

From the last identity we obtain

$$\begin{aligned}
1+2^2+\dots+2^{k+1} &= 2^{k+1}+2^{k+1}-1\\
&= 2\cdot 2^{k+1}-1\\
&= 2^{(k+1)+1}-1
\end{aligned}$$

Therefore, we have $k+1\in A$.

At the end, we have:

① $1\in A$

② $k\in A \Rightarrow k+1\in A$.

Together, ① and ② imply that $A=\mathbb{N}$. That is, our identity holds for every natural number.

Problem ② (Sum of geometric progressions)

Use mathematical induction to show that

$$a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r-1}$$

for every $r \neq 1$, $n \in \mathbb{N}$ and $a \in \mathbb{R}$. //

Solution: Observe that the formula holds for $n=1$, all $r \neq 1$ and all $a \in \mathbb{R}$:

$$\begin{aligned} a + ar &= a \cdot (r+1) \\ &= \frac{a \cdot (r+1)}{r-1} \cdot (r+1) \\ &= \frac{a(r^2-1)}{r-1} \\ &= \frac{ar^2 - a}{r-1} \end{aligned}$$

Define

$$A = \left\{ n \in \mathbb{N} : a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r-1}, \forall r \neq 1, \forall a \in \mathbb{R} \right\}$$

Note that $1 \in A$. This is called the base of the induction.

Let $k \in A$ be an element in A. Next, we show that $k+1 \in A$. This is called the inductive step.

If $k \in A$, then

$$a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a}{r-1} + ar^{k+1} \quad \forall r \neq 1, \forall a \in \mathbb{R}.$$

here we used that $k \in A$.

From the last identity we obtain

$$a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a + ar^{k+1}(r-1)}{r-1}$$

$$= \frac{\cancel{ar^{k+1}} - a + ar^{(k+1)+1} - \cancel{ar^{k+1}}}{r-1}$$

$$= \frac{ar^{(k+1)+1} - a}{r-1} \quad \forall r \neq 1, \forall a \in \mathbb{R}$$

Therefore, we have $k+1 \in A$.

At the end, we have:

① $1 \in A$

② $k \in A \Rightarrow k+1 \in A$.

Together, ① and ② imply that $A = \mathbb{N}$. That is, our identity holds for every natural number.

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Problem ③: For $i \in \mathbb{N}$, define $H_i := 1 + \frac{1}{2} + \dots + \frac{1}{i}$.

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2}.$$

Solution: Observe that the formula holds for $n = 1$:

$$H_{2^1} = 1 + \frac{1}{2} \geq 1 + \frac{1}{2}.$$

Define

$$A = \left\{ n \in \mathbb{N} : H_{2^n} \geq 1 + \frac{n}{2} \right\}.$$

Note that $1 \in A$. This is called the base of the induction.

Let $k \in A$ be an element in A. Next, we show that $k+1 \in A$. This is called the inductive step.

If $k \in A$, then

$$H_{2^{k+1}} = 1 + \frac{1}{2} + \dots + \frac{1}{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}}$$

$$\geq 1 + \frac{k}{2} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}} \qquad (*)$$

In the last inequality, we used that $k \in A$.

Now, observe that $\frac{1}{i} \geq \frac{1}{j}$ for all $i \leq j$. Therefore,

$$\frac{1}{i} \geq \frac{1}{2^{k+1}} \quad \text{for all } i \in \{2^k+1, 2^k+2, \dots, 2^{k+1}\} \qquad (**)$$

Note that the set $\{2^k+1, 2^k+2, \dots, 2^{k+1}\}$ has $2^{k+1} - (2^k+1) + 1 = 2^k$ elements (here we are using the general fact that $\#\{a, \dots, b\} = b - a + 1$)

Therefore, $(**)$ together with this observation gives

$$\frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}} \geq \frac{1}{2^{k+1}} \cdot \#\{2^k+1, 2^k+2, \dots, 2^{k+1}\}$$

$$\geq \frac{1}{2^{k+1}} \cdot 2^k$$

$$= \frac{1}{2}. \qquad (***)$$

Combining (*) with (***) we obtain

$$H_{2^{k+1}} \geq 1 + \frac{k}{2} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}} \qquad (\text{here we used } (*))$$

$$\geq 1 + \frac{k}{2} + \frac{1}{2} \qquad (\text{here we used } (***))$$

$$= 1 + \frac{k+1}{2}.$$

Therefore, we have $k+1 \in A$.

At the end, we have:

① $1 \in A$

② $k \in A \Rightarrow k+1 \in A$.

Together, ① and ② imply that $A = \mathbb{N}$. That is, our identity holds for every natural number. //

Definition: Let $a, b \in \mathbb{N}$. We say that b is divisible by a, or that a divides b if there exists $q \in \mathbb{N}$ such that $b = a \cdot q$.

Notation: $a \mid b$.

Problem ④ Show that $3 \mid n^3 - n$ for every $n \in \mathbb{N}$.

Solution: Observe that $3 \mid 1^3 - 1$.

Define

$$A = \{ n \in \mathbb{N} : 3 \mid n^3 - n \}.$$

Note that $1 \in A$. This is called the base of the induction.

Let $k \in A$ be an element in A. Next, we show that $k+1 \in A$. This is called the inductive step.

Note that

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$$

$$= (k^3 - k) + 3 \cdot (k^2 + k) \qquad (*)$$

As $k \in A$, there exists $q = q(k)$ such that $k^3 - k = 3q$.

From $(*)$, it follows that

$$(k+1)^3 - (k+1) = 3q + 3 \cdot (k^2 + k)$$

$$= 3 \cdot (q + k^2 + k).$$

By the last equality, we conclude that there exists $s \in \mathbb{N}$ such that $(k+1)^3 - (k+1) = 3s$. (in this case, s is $q + k^2 + k$). Thus, it follows that $k+1 \in A$.

At the end, we have:

1. $1 \in A$
2. $k \in A \Rightarrow k+1 \in A$.

Together, 1 and 2 imply that $A = \mathbb{N}$. That is, our identity holds for every natural number.

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