LOGIC OPERATORS :

We can combine propositions by using logic connectives

We first introduce what is negation. Then, we introduce four logic connectives:

- conjunction
- disjunction
- conditional
- biconditional.

__Definition:__ The negation of p is the statement "it is not the case that p".

__Notation__ : $\neg p$. (not p).

If p is false, then $\neg p$ is true.

If p is true, then p is false.

p	$\neg p$
T	F
F	T

he negation of a proposition can be considered the result of the operation of the __negation operator__ on a proposition.

__Definition__ (Conjunction)

Let p and q be propositions.

The <u>conjunction</u> of p and q is the proposition "p and q". It is true when both p and q are true and it is false otherwise.

<u>Notation</u>: $p \wedge q$ (the symbol $\wedge$ resembles $\cap$)

<u>Definition</u> (Disjunction)

Let p and q be propositions.

The <u>disjunction</u> of p and q is the proposition "p or q". It is false when both p and q are false and it is true otherwise.

<u>Notation</u>: $p \vee q$ (the symbol $\vee$ resembles $\cup$)

<u>The truth table</u>:

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

<u>Conditional statements</u>:

Definition: Let p and q be propositions. The conditional statement $p \to q$ is the proposition "If p, then q". The conditional statement is false when p is true and q is false, and true otherwise. The proposition p is called hypothesis (premise) and q is called conclusion (consequence).

Obs: conditional statements are also called implications.

Other conditional statements:

Let $p \to q$ be a conditional statement. We say that

- $q \to p$ is the converse of $p \to q$
- $\neg q \to \neg p$ is the contrapositive of $p \to q$
- $\neg p \to \neg q$ is the inverse of $p \to q$.

Definition (biconditional)

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q".

$p \leftrightarrow q$ is true when p and q have the same true values, and it is false otherwise.

Example:

Find the truth table of $(p \vee \neg q) \rightarrow (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Propositional equivalences

Definition: A compound proposition is an expression formed from propositional variables using logical operators.

We may think that a compound prosition is a function whose input is a tuple of propositions an whose output is a proposition obtained from the input by logical connectives.

Definition: A compound proposition that is always true, no matter what the true values of the propositional variables that occur to it, is called tautology

A compound proposition that is always false is called a contradiction.

Examples: $p \vee \neg p$ is a tautology.

$p \wedge \neg p$ is a contradiction.

Definition (Logical equivalences)

Compound propositions p and q that have the same true values in all possible cases are called logically equivalent. That is, $p \leftrightarrow$ is a tautology.

Notation: $p \equiv q$ or $\Leftrightarrow$.

Obs: The symbol $\equiv$ is not a logical connective.

and "$p \equiv q$" is not a compound proposition.

"$p \equiv q$" is the statement "$p \leftrightarrow q$ is a tautology".

De Morgan laws:

we have ① $\neg(p \wedge q) \equiv \neg p \vee \neg q$

② $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

proof: We will prove only ② using the truth table.

② can be proven in a similar way.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F

T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The columns are equal!

There are some important logical equivalences.

Let **T** denote the compound proposition that is always true and let **F** denote the compound proposition that is always false.

Identity law: $p \wedge \mathbf{T} \equiv p$

Double negation: $\neg(\neg p) \equiv p$

Distributive law:

- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

<u>Exercise</u>: Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

<u>Exercise</u>: Show that $p \to q \equiv \neg p \vee q$.

<u>Exercise</u>: Show that $(p \wedge q) \to (p \vee q)$.