Claim: If $f: A \to B$ is invertible, then f admits an unique inverse function.

Proof: Let $g_1 : B \to A$ and $g_2 : B \to A$ be inverse functions of f. By the previous lemma, g_1, g_2, f are bijective functions.

Thus,

$$f(g_1(b)) = b = f(g_2(b))$$

As f is bijective, we have

$$g_1(b) = g_2(b)$$

for every $b \in B$. $\blacksquare$

Notation: When f is a bijective function, we denote by f^{-1} its unique inverse function.

Examples:

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$: $f^{-1}(x) = x$

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+1$:

$$x = g(g^{-1}(x)) = 2g^{-1}(x) + 1$$

This gives $\frac{x-1}{2} = g^{-1}(x)$.

<u>Obs</u>: Let $f: A \to B$ be a function.

Suppose there exists a function $g: B \to A$ such that $f(g(b)) = b \ \forall b$. Not necessarily we have $g(f(a)) = a \ \forall a$.

<u>Example</u>: $f: \mathbb{R} \to [0,\infty)$, $f(x) = |x|$

$g: [0,\infty) \to \mathbb{R}$, $g(x) = x$

Then, $f(g(x)) = f(x) = x$.

But $g(f(x)) = g(|x|) = |x|$.

In particular, $g(f(-1)) = 1 \neq -1$.

<u>Definition</u> (Square-root function)

The square-root function is the inverse function of $f: [0,\infty) \to [0,\infty)$, $f(x) = x^2$.

In particular, $\sqrt{(-3)^2} = -3$ is FALSE.

USEFUL INEQUALITIES:

Triangle inequality: If $x, y \in \mathbb{R}$, then $|x+y| \leq |x|+|y|$.

proof: Note that $xy \leq |x||y|$.

Then,
$$\underbrace{x^2 + 2xy + y^2} \leq |x|^2 + 2|x||y| + |y|^2$$

$$(x+y)^2 \leq (|x|+|y|)^2$$

$$\Downarrow$$

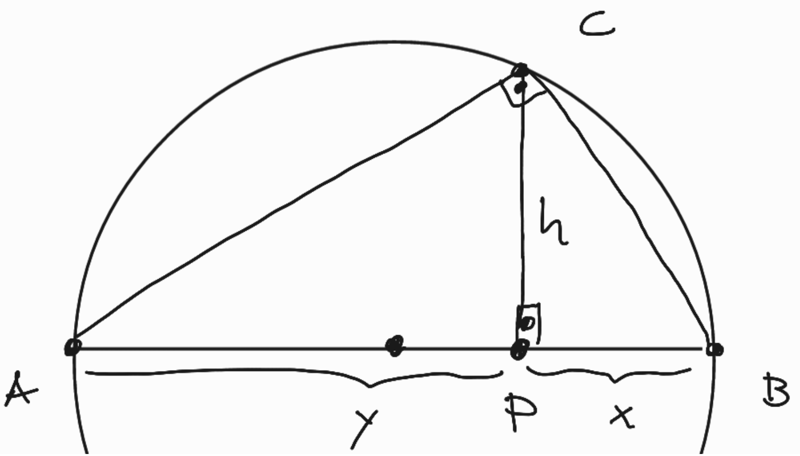
$$|x+y| \leq \big||x|+|y|\big| = |x|+|y|.$$

$\blacksquare$

Arithmetic-Geometric mean inequality (AM-GM):

If $x, y \in \mathbb{R}_{\geq 0}$, then $\dfrac{x+y}{2} \geq \sqrt{xy}$.

proof:



By similarity: $\frac{h}{x} = \frac{y}{h} \Rightarrow h = \sqrt{xy}$

$$\text{Ratio} = \frac{x+y}{2} \geq h = \sqrt{xy}.$$

$\square$

<u>Another proof:</u>

$$\left(\frac{x+y}{2}\right)^2 \geq xy \iff x^2 + 2xy + y^2 \geq 4xy$$

$$\iff (x-y)^2 \geq 0.$$

$\blacksquare$

<u>LANGUAGE AND PROOF:</u>

<u>Definition</u> (Proposition, statement)

A proposition (or statement) is a declarative sentence that is either true or false, but not both.

Examples of propositions:

- today it is raining
- Toronto is the capital of Brazil
- $1+1=2$

Examples of non-propositions:

- What time is it?
- $x+1=2$

Why? The 1st is not declarative.
The 2nd is neither true or false.

Notation: we use letters to represent propositions
These are called propositional variables.
It is similar to the use of letters to denote a numerical value.

Notation: T denotes true
F denotes false