Let $A \subseteq \mathbb{R}$. We say that $f: A \to \mathbb{R}$ is

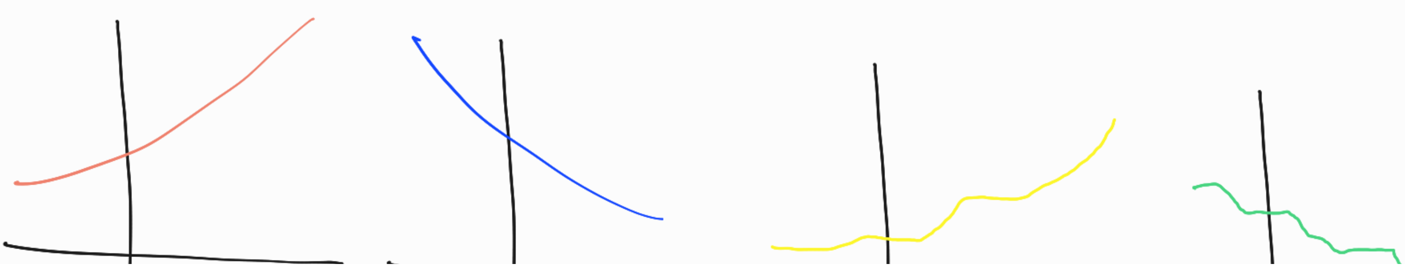
increasing
decreasing
non-decreasing
non-increasing

if the following holds for every $x < x'$ with $x \in A$ and $x' \in A$:

$$f(x) < f(x')$$
$$f(x) > f(x')$$
$$f(x) \leq f(x')$$
$$f(x) \geq f(x')$$

How the graph of these functions look like?

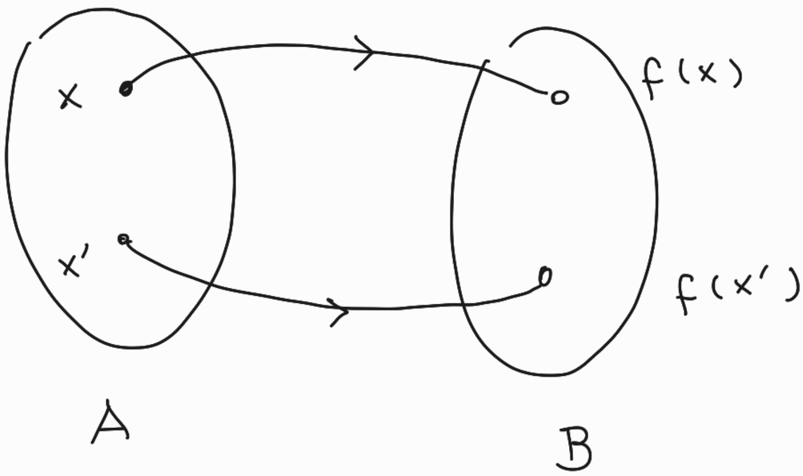
Examples:



increasing decreasing non-decreasing non-increasing

<u>Definition</u> (Injective, one-to-one)

$f : A \to B$ is injective or one-to-one if for every $x \in A$ and $x' \in A$ with $x' \neq x$ we have $f(x) \neq f(x')$.



In other words, two distinct elements in A do not have the same image.

<u>Examples</u>:

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$ is injective

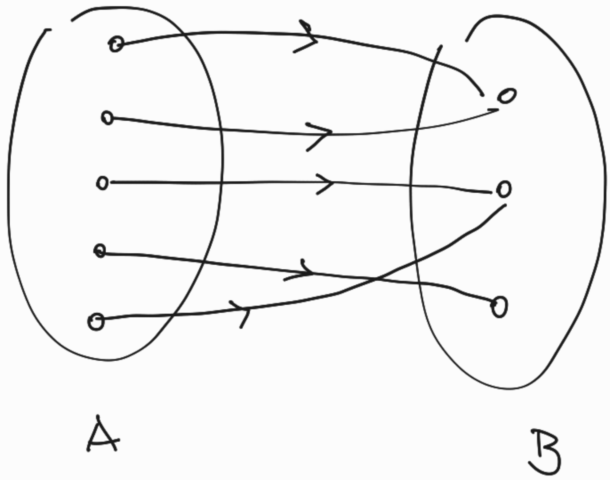
$g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ is not injective

Definition (

(surjective, onto)

$f : A \to B$ is a surjective or onto function if for $b \in B$ there exists $a \in A$ such that $f(a) = b$.

In other words, $\text{Image}(f) = B$.



Examples:

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 2$ is surjective

$g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ is not surjective.

<u>Definition</u> (bijection)

$f : A \to B$ is a bijection if f is injective and surjective.

<u>Examples</u>:

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$ is a bijection

$g : \mathbb{R} \to \mathbb{R}$, $g(x) = (x-1)(x+1)$ is not a bijection.

<u>Definition</u> (Identity function)

The identity function on a set S is a function $f : S \to S$ defined by $f(x) = x \quad \forall x \in S$.

<u>Composition of functions</u> :

Let $f : A \to B$ and $g : C \to A$ be functions.

The function $f \circ g : C \to B$ defined by $(f \circ g)(x) = f(g(x))$ is the composition of f with g.

<u>Examples</u> :

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x+1$

$g : \mathbb{R} \to \mathbb{R}$, $g(x) = 6x+7$

$h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^2$

We have: $f(g(x)) = f(6x+7) = 2(6x+7)+1$

$$= 12x + 15$$

$$f(h(x)) = f(x^2) = 2x^2+1$$

$$h(f(x)) = h(2x+1) = (2x+1)^2$$

$= 4x + 4x + 1$

Note that $h \circ f \neq f \circ h$ in this example.

Definition (Invertible function)

$f : A \to B$ is invertible if there exists a function $g : B \to A$ such that $f(g(b)) = b$ for every $b \in B$

$g(f(a)) = a$ for every $a \in A$.

We call g an inverse function of f.

Lemma: $f : A \to B$ is invertible $\Leftrightarrow$ f is bijective.

proof:

($\Rightarrow$) Suppose that $f : A \to B$ is invertible and let g be an inverse function of f.

As $f(g(b)) = b$, we have $\text{Image}(f) = B$.

This implies that f is surjective.

In the same way, as $g(f(a)) = a$, we have that g is surjective.

g surjective $\Rightarrow$ For every $a \in A$ there is a $b \in A$ $\underbrace{\text{such that } g(b) = a}_{(*)}$.

We now use this property

to show that f is injective

Let $a, a' \in A$ be such that $f(a) = f(a')$.

By property $(*)$, there exists b and b'

such that $g(b) = a$ and $g(b') = a'$.

Then, we have

$$b = f(g(b)) = f(a) = f(a') = f(g(b')) = b'$$

Therefore, $b = b'$. This implies

$$a = g(b) = g(b') = a'.$$

That is, we have shown that

if $f(a) = f(a')$, then $a = a'$.

Therefore, f is injective!

$(\Leftarrow)$ If f is bijective, we define $g: B \to A$

as follows:

For $b \in B$, let a be the unique

element in A for which $f(a) = b$.

Define $g(b) = a$.

By definition, $f(a) = f(g(b)) = b$.

$g(f(a)) = g(b) = a$.

$\blacksquare$