Sets

Definition (Set): A set is a collection of distinct objects. The empty set, denoted by $\emptyset$ or $\{\}$, is the unique set with no elements.

Examples:

- $\{1,2,3\}$ • $\{A,B,C,D\}$ • $\{Ana, Bob\}$ • $\{2, a, Fred\}$
- $\mathbb{R}$ (real numbers)

Definition (Elements):
The objects in a set are its elements or members
If an object x belongs to a set S, we denote $x \in S$
(read: x belongs to S, or x in S).
Otherwise, we denote $x \notin S$.
(read: x does not belong to S, or x is not in S).

Examples:

- $2 \in \{1,2,3\}$
- $3.14 \in \mathbb{R}$
- $4 \notin \{1,2,3\}$

Definition (Subset): We say that a set A is a subset of a set B if every element of A belongs to B.

Notation:

$A \subseteq B$ (read : A is contained in B)

or

$B \supseteq A$ (read : B contains A)

$A \subseteq B$ and $B \supseteq A$ denote exactly the same thing!

When A is not contained in B, we denote $A \nsubseteq B$ or $B \nsupseteq A$.

Examples : • $\{1\} \subseteq \{1,2,3\}$ • $\{1,3\} \subseteq \{1,2,3\}$ • $\mathbb{Q} \subseteq \mathbb{R}$.

Definition (Equal sets).

We say that a set A is equal to a set B if they have the same elements. That is, $A \subseteq B$ and $B \subseteq A$.

Notation : $A = B$.

We say that two sets are distinct if they are not equal. Notation : $A \neq B$.

Definition (Proper subsets)

We say that a set A is a proper subset of a set B if $A \subseteq B$ but $A \neq B$. Notation : $A \subsetneq B$.

Examples :

$S = \{\text{Kansas, Kentucky}\}$

$T = \{\text{States in the US whose names begin with K}\}$

We have $S = T$.

Subsets of S:

- $\emptyset \subseteq S$
- $\{\text{Kentucky}\} \subseteq S$
- $\{\text{Kansas}\} \subseteq S$
- $\{\text{Kansas, Kentucky}\} \subseteq S$.

Obs: As sets are collections of distinct objects, the set $\{1, 3, 3, 5, 5\}$ is equal to $\{1, 3, 5\}$.

In the previous example; you might not be convinced why $\emptyset \subseteq S$. We show this in the next lemma.

Lemma: Let S be a set. Then, $\emptyset \subseteq S$.

Proof: The proof is by contradiction. Suppose that $\emptyset \nsubseteq S$. Then, there must be an element $x \in \emptyset$ such that $x \notin S$. This is a contradiction, as $\emptyset$ has no elements. $\blacksquare$

This black square says that we finished our proof.

Definition (Power set): the power set of a set A is the collection of all subsets of A.

Notation: 2^A or $\mathcal{P}(A)$.

In the previous example, $S = \{\text{Kansas, Kentucky}\}$ and the power set is

$$2^S = \{ \quad \}$$

$C = \{\emptyset, \{\text{Kansas}\}, \{\text{Kentucky}\}, \{\text{Kansas}, \text{Kentucky}\}\}$.

Notation (specifying a set):

Given a set A, we write $\{x \in A : \text{condition}(x)\}$ to mean the set of x in A such that x satisfies condition(x).

Example:

- $S = \{x \in \mathbb{R} : x^2 + 2x + 1\}$

  S is the set of real numbers satisfying the equation $x^2 + 2x + 1 = 0$.

- Let $S = \{x \in \mathbb{R} : x^2 < x\}$ and

  $T = \{x \in \mathbb{R} : 0 < x < 1\}$.

  Claim: $S = T$

  proof:

  First, we show that $T \subseteq S$: let $x \in T$. Then,

  $$x > 0 \text{ and } x < 1 \implies x \cdot x < x$$

  we multiplied this ineq. by x on both sides.

  Thus, $x^2 < x$. This implies $x \in S$.

  We conclude that every element in T is an element of S.

Now, we show that $S \subseteq T$: let $x \in S$.

We have:

$$0 \leq x^2 < x \Rightarrow x > 0.$$

As $x > 0$, we can multiply by $\frac{1}{x}$ both sides of $x^2 < x$. This gives $x < 1$.

Therefore, $0 < x < 1$, and hence $x \in 1$.

We conclude that every element in S is an element of T.

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