## Definition (Cardinality)

Let $S$ be a set. If $S$ has exactly $n$ distinct elements, where $n \in \mathbb{N}$, we say that $S$ is a finite set and that $n$ is the cardinality of $S$. Notation: $|S|$ or $\# S$.

Examples:

- $|\phi| = 0$
- $A = \{\text{odd numbers between } 0 \text{ and } 10\}$

  $|A| = 5$
- $S = \{x \in \mathbb{R} : x^2 - 2x + 1 = 0\}$

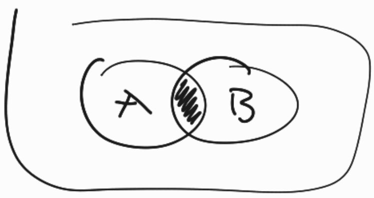
  $|S| = 1$

## Inclusion-Exclusion principle for two sets:

Let $A$ and $B$ be two finite sets. Then,

$$|A\cup B| = |A| + |B| - |A\cup B|.$$

Intuition:




The sum $|A| + |B|$ counts the elements in the intersection $A\cap B$ twice.

proof:

We can write

$$A\cup B = \underbrace{(A\setminus B) \cup (B\setminus A) \cup (A\cap B)}$$

Observe that this is a disjoint union (prove it!).

It follows that

$$|A \cup B| = |A \setminus B| + |B \setminus A| + |A \cap B|.$$

$$= \underbrace{|A \setminus B| + |A \cap B|}_{=|A|} + \underbrace{|B \setminus A| + |A \cap B|}_{=|B|} - |A \cap B|$$

$$= |A| + |B| - |A \cap B|.$$

We now move towards the topic <u>functions</u>. But before that we need to establish some definitions and notation.

<u>Definition</u>: The ordered $n$-tuple $(a_1, \ldots, a_n)$ is the ordered collection that has $a_1$ as its first element, ..., $a_n$ as its $n$th element.

<u>Definition</u>: We say that two ordered $n$-tuples are equal if each corresponding pair of their elements is equal. That is, $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if $a_i = b_i$ for every $i \in [n]$.

<u>Example</u>: • $(1,2,3) \neq (3,1,2)$ • $(1,2,3,3,4) \neq (1,3,3,2,4)$
• $(1,2,3,1) \neq (1,1,1,2)$ • $(1,2,3) \neq (1,2,3,4)$.

<u>Definition</u> (Cartesian product): Let $A$ and $B$ be sets. The cartesian product of $A$ and $B$, denoted by $A \times B$, is the set of all ordered pairs $(a,b)$ with $a \in A$ and $b \in B$. Thus,

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}.$$

(read: A cartesian product B; A times B, A product B)

<u>Notation</u> (Intervals): let $a$ and $b$ be real numbers, with $a \leq b$. We write:

- $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

  (closed interval)

- $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$

  (closed on the left, open on the right)

- $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$

  (closed on the right, open on the left)

- $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

  (open interval).

<u>Obs</u>: Do not get confused. We use the same notation for the open interval $(a,b)$ and for the ordered pair (2-tuple) $(a,b)$.

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## <u>Functions</u>

<u>Definition</u> (Function): Let $A$ and $B$ be non-empty sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$.

For a function $f$ from $A$ to $B$, we write $f : A \to B$.

The set A is called domain and the set B is called codomain of f.

Notation: $f(a)$ is the assignment of f to a.

$Co(f)$ = Codomain of f.

$Do(f)$ = Domain of f.

The image of a function $f: A \to B$ is the set $\{f(a) : a \in A\}$. Notation: $Im(f)$ or $f(A)$.

Obs: function = map = transformation.

Obs: $f(x)$ denotes an element of the codomain. It does not denote a function.

Usually, functions are given by formulas.

Examples:

① $S: \mathbb{R} \to \mathbb{R}$, $S(x) = x^2$.

Domain: $\mathbb{R}$, Codomain: $\mathbb{R}$, Image: $\mathbb{R}_{\geq 0}$.

$\{x \in \mathbb{R} : x \geq 0\}$

② $T: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $T(x,y) = x+y$.

Domain: $\mathbb{R} \times \mathbb{R}$, Codomain: $\mathbb{R}$, Image: $\mathbb{R}$

proof: • Image $\subseteq \mathbb{R}$: $x+y$ is alway a real number

• $\mathbb{R} \subseteq$ Image: let $r \in \mathbb{R}$. We have $f(0,r) = r$

③ $f: \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $f(x) = (x, x^2)$.

Domain: $\mathbb{R}$, Codomain: $\mathbb{R} \times \mathbb{R}$

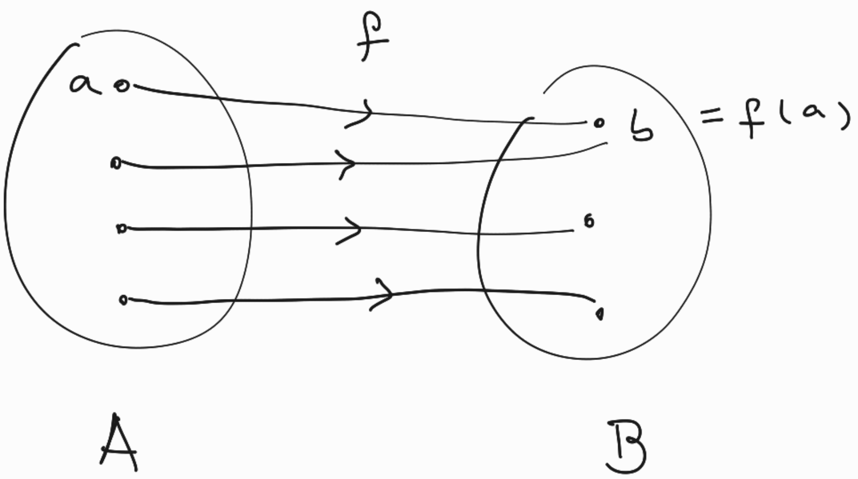
Image : $\{ (x, x) : x \in \mathbb{R} \}$.

__Obs__ : note that $\text{Image}(f) \neq \mathbb{R} \times \mathbb{R}_{\geq 0}$.

__Definition__ (well-defined): We say that $f : A \to B$ is well-defined if the rules assign to each element of $A$ exactly one element of $B$.

## Schematic representation :

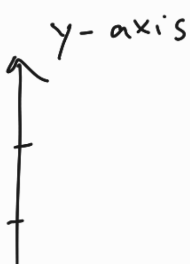
A function $f : A \to B$ can be schematically represented as follows :

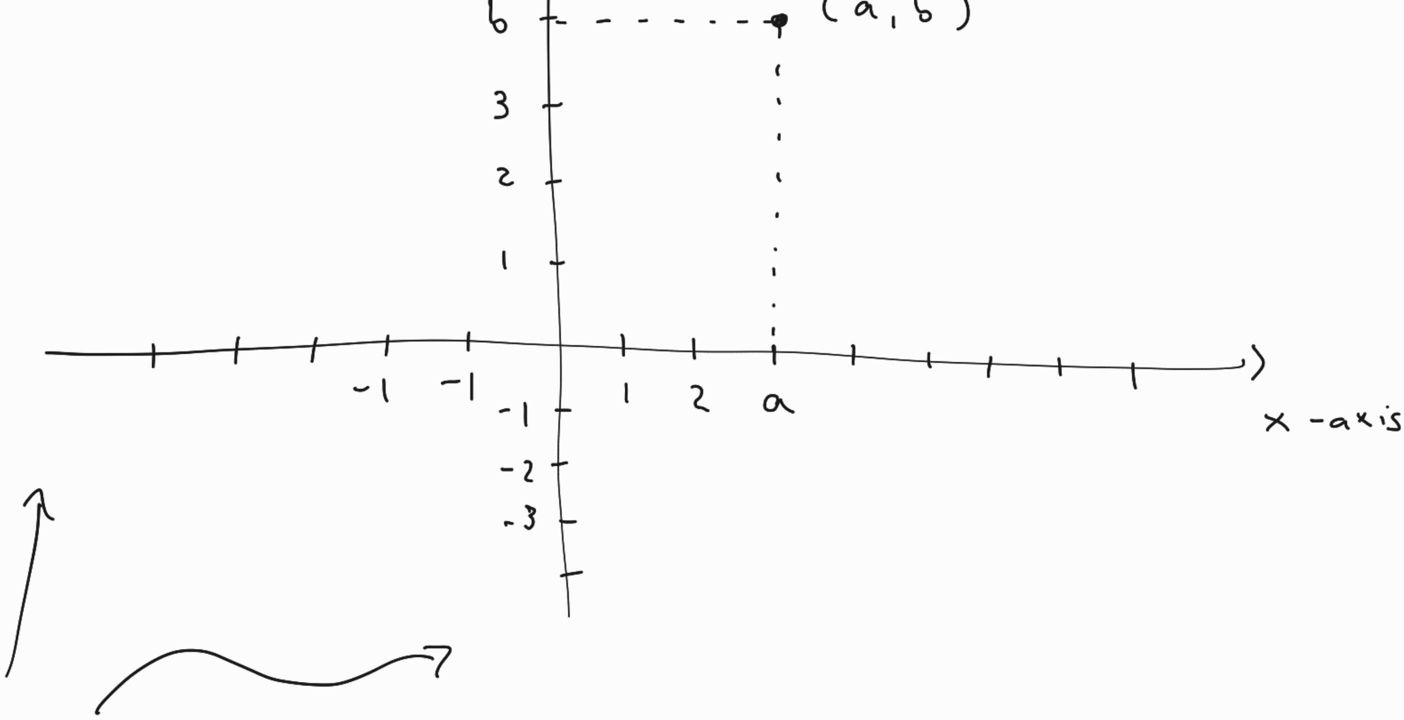



## Graphic representation :

__Definition__ (Cartesian plane)

The cartesian plane is a representation of the points of $\mathbb{R} \times \mathbb{R}$ in the plane as follows :






these are perpendicular real lines that meet in (0,0).

<u>Definition</u> (graph of a function) Let $f : A \to B$ be a function with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. The graph of $f$ is the set of points $\{(a, f(a)) : a \in \mathbb{R}\}$ represented in the cartesian plane.

<u>Obs</u>:

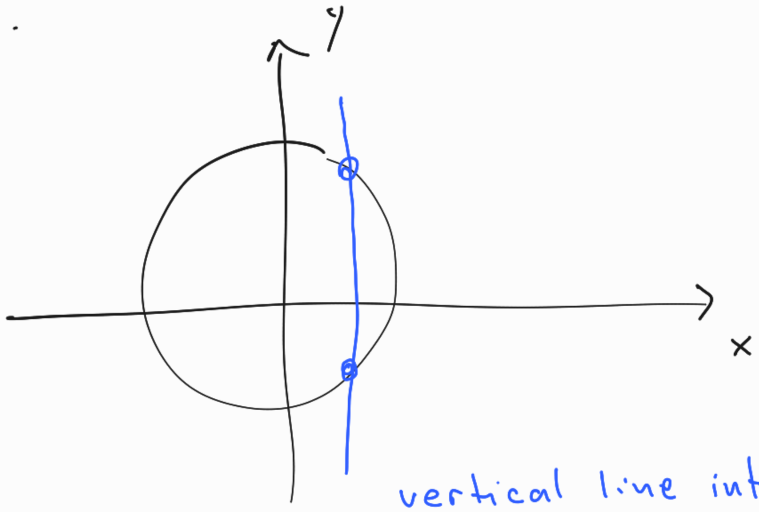
A non-empty set of points $S$ in the plane is the graph of a function if and only if for each $x \in \mathbb{R}$, the vertical line passing through $(x, 0)$ contains at most one element element of $S$.

<u>Example</u>:

- Let $f : [-1, 1] \to \mathbb{R}$ be defined by $(f(x))^2 + x^2 = 1$.

well-defined. Indeed, when $x = 0$, $f(0) = 1$ and $f(0) = -1$ are both valid choices.

- The graph of a circle cannot be the graph of a function.




vertical line intersects two points.

<u>Definition</u> (Real-valued functions).

We say that $f$ is a real-valued function if its image is a subset of $\mathbb{R}$.

<u>Notation</u>: Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions. We denote:

- $f + g : A \to \mathbb{R}$, $(f+g)(x) = f(x) + g(x)$
- $f \cdot g : A \to \mathbb{R}$, $(f \cdot g)(x) = f(x)\, g(x)$.
- If $g(x) \neq 0 \ \forall x \in \mathbb{R}$, we denote by $f/g : A \to \mathbb{R}$ the function defined by $(f/g)(x) = f(x)/g(x)$.

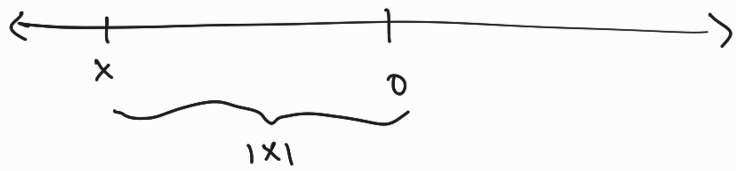
<u>Definition</u> (Absolute value/Modulus).

Let $x \in \mathbb{R}$. We define the absolute value of $x$, denoted by

$|x|$, to be

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

$|x|$ can be seen as the distance from $x$ to $0$ in the real line:




<u>Definition</u> (Bounded set): We say that a set $S \subseteq \mathbb{R}$ is bounded if there exists $M \in \mathbb{R}$ such that $|x| \leq M$ for all $x \in S$. Otherwise, we call $S$ unbounded.

<u>Definition</u> (Bounded function).

Let $f : A \to B$ be a real-valued function.
We say that $f$ is bounded if the set $f(A)$ is bounded.

<u>Examples</u>:

① $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1$ is bounded.

② $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{otherwise} \end{cases}$ is bounded.

③ $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x$ is not bounded.