Notation : $\mathbb{N} = \{1, 2, 3, \ldots\}$.

The penny problem

Given n piles of pennies (no order between the piles), do the following move :

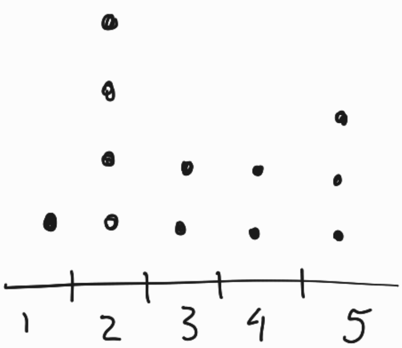
remove one coin from each pile to make a new pile.

Let S_n be the set of lists of n piles that do not change after this move.

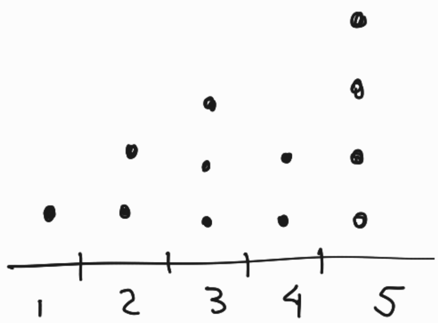
Problem : Describe the set S_n for every $n \in \mathbb{N}$.

Let us understand the problem through a small example.

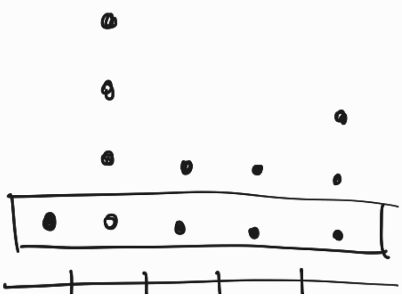
For us, the configurations of 5 piles



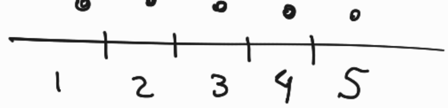
and



is THE SAME. That is, order does not matter.



after one move



Describing the set S_n:

Let a be a list of n piles such that $a \in S_n$.
Denote by b the list obtained from a after one move.
As $a \in S_n$, we have $b = a$.

We start by showing that a has a list with one coin.

Claim 1: a has a list with one coin.

proof: Suppose for contradiction that every list in a has at least two coins.

Then, after one move we would have $n+1$ piles.
This implies $a \neq b$, a contradiction $\boxtimes$

Claim 2: Let $i \in \mathbb{N}$. If b has a pile of size i, then a has a pile of size $i+1$.

proof: the claim follows from the fact that we remove one element from each pile to obtain b. $\boxtimes$

Let us put claims 1 and 2 together:

$a \in S_n \underset{\text{(claim 1)}}{\Rightarrow} \exists$ list of size 1 in a and b

If $n > 3$ we have

$\exists$ list of size 1 in a and b $\Longrightarrow$ $\exists$ list of size 2 in a and b.

(claim 2)

If $n \geq 3$, we have

$\exists$ list of size 2 in a and b $\Longrightarrow$ $\exists$ list of size 3 in a and b.

(claim 2)

and so on...

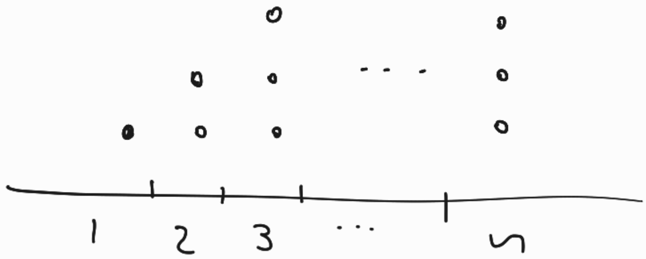
By induction, we infer from these sequences of implications that

$a \in S_n \Rightarrow$ a contains a list of size i for every $i \in \{1, \dots, n\}$.

As a has n lists, it follows that a should contain exact one list of size i, for every $i \in \{1, \dots, n\}$.

We conclude that S_n has one element.

This element can be represented grafically as



Notation :

- $\mathbb{N} = \{1, 2, \dots\}$
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$

  (This is not a very precise definition. But we will assume that you are confortable with the concept of fraction.)

- $\mathbb{R}$ = set of real numbers

  (It cannot be easily described with the tools we have. We will see a precise definition at the end of this course, as well as for $\mathbb{Q}$).

- $\{a, \dots, b\} = \{i \in \mathbb{Z} : a \leq i \leq b\}$
- $[n] = \{1, 2, \dots, n\}$
- Even numbers $= \{2k : k \in \mathbb{Z}\}$
- Odd numbers $= \{2k+1 : k \in \mathbb{Z}\}$.

Obs : 0 is an even number.

Definition (Set operations)

Let A and B be sets.

$A \cup B$ = all elements in A or B.

(read : A union B)

$A \cap B$ = all elements in A and B.

(read : A intersection B)

$A \setminus B$ = all elements in A but not in B.

The notation $A - B$ can also be used.

(read : A minus B)

<u>Definition</u> (disjoint sets)

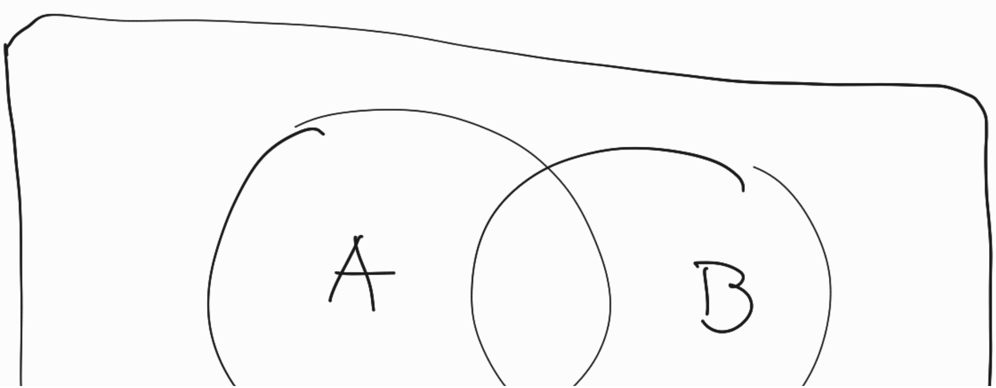
We say that the sets A and B are disjoint if $A \cap B = \emptyset$.

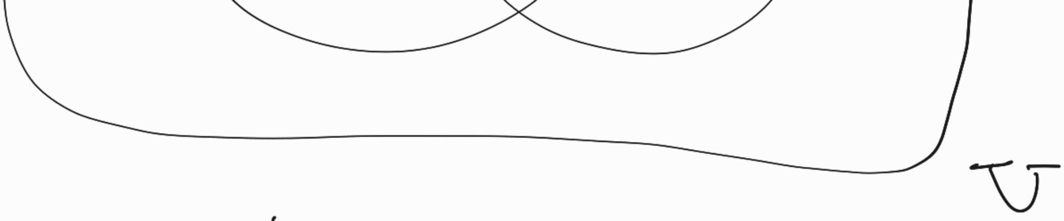
<u>Definition</u> (Complement)

If a set A is contained in some universe U, then the complement of A is the set of elements in U but not in A.

<u>Notation</u> : A^c or $\bar{A}$. (read : A complement).

<u>Venn diagram</u> :





From the Venn diagram we can infer some useful identities. However, we need to prove them formally.

A very useful identity is given by the next lemma:

Lemma: Let A and B be sets. Then,
$(A \cup B)^c = A^c \cap B^c$.

proof:

(1) First, we show that every element in $A^c \cap B^c$ is contained in $(A \cup B)^c$.

Suppose that $A^c \cap B^c$ is non-empty.
Let $x \in A^c \cap B^c$. We have

$$x \in A^c \cap B^c \Rightarrow x \in A^c \text{ and } x \in B^c$$
$$\Rightarrow x \notin A \text{ and } x \notin B$$
$$\Rightarrow x \notin A \cup B$$
$$\Rightarrow x \in (A \cup B)^c$$

We conclude that $x \in (A \cup B)^c$.

(2) Now, we show that every element in $(A \cup B)^c$ is contained in $A^c \cap B^c$.

Suppose that $(A \cup B)^c$ is non-empty

Let $x \in (A \cup B)^c$. We have

$$x \in (A \cup B)^c \Rightarrow x \notin A \cup B$$
$$\Rightarrow x \notin A \text{ and } x \notin B$$
$$\Rightarrow x \in A^c \text{ and } x \in B^c$$
$$\Rightarrow x \in A^c \cap B^c.$$

We conclude that $x \in A^c \cap B^c$.

① + ② imply that we cannot have $A^c \cap B^c \neq \phi$ and $(A \cup B)^c = \phi$, as $A^c \cap B^c \subseteq (A \cup B)^c$.

Similarly, we cannot have $(A \cup B)^c \neq \phi$ and $A^c \cap B^c = \phi$, because $(A \cup B)^c \subseteq A^c \cap B^c$.

Our first conclusion is: If $A^c \cap B^c = \phi$, then $(A \cup B)^c = \phi$, and hence $A^c \cap B^c = (A \cup B)^c$.

Our second conclusion is: If $A^c \cap B^c \neq \phi$, then $(A \cup B)^c \neq \phi$, and hence ① + ② imply $A^c \cap B^c = (A \cup B)^c$.

$\blacksquare$